SCMS,

Notes for 2018 fall AG program in SCMS

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2019 Spring

These notes are taken from talks and lectures in 2018 AG Program at SCMS, covering several central topics in AG.

Contents

0.1 Topic 1 : Cohomological perspective on Moduli space

0.1.1 Prof.Rahul Pandharipande's Distinguished lectures

Moduli of K3 surfaces and Lehn's conjecture and further developments

Lecture 1:

Let $\mathcal{S} \to \mathcal{M}$ be a family of surface. eg,

- $\mathcal{M} = pt$,
- $\mathcal{M}$ = moduli space of q-p K3 of fixed degree, $\mathcal{S}$ = universal K3.
- $\mathcal{M} = M_g \times M_h$, $\mathcal{S} = C_g \times C_h$

let $\mathcal{L} \to \mathcal{S}$ be a line bundle, then one can define family of tautological bundles

$$\begin{array}{ccc} \mathcal{L}[n] & \longrightarrow & \mathcal{S}[n] \\ & & \big\downarrow p \\ & & \mathcal{M} \end{array} \tag{0.1}$$

$\forall Z \in \mathcal{S}[n]$, $Z_x \subset \mathcal{S}_x$ is a subscheme of length n for any $x \in \mathcal{M}$. The fiber of $\mathcal{L}[n]$ at Z is just

$$H^0(Z, \mathcal{L}|_Z)$$

Globally construction of $\mathcal{L}[n]$ will use universal subscheme

$$\mathcal{S} \times \mathcal{S}[n] \supset U[n] \quad \xrightarrow{p_1} \mathcal{S}, \quad \xrightarrow{p_2} \mathcal{S}[n]$$

and $\mathcal{L}[n] := (p_1)_*(\mathcal{O}_{U[n]} \otimes p_2^*\mathcal{L})$

Q: what is $p_*(s(\mathcal{L}[n])) \in A^*(\mathcal{M})$

1. For $= pt$, it is equivalent to study $\int_{S[n]} s_{2n}(L[n]) =?$

The known result so far

1. M.Lehn 1999
2. MOP 2015, C.Voisin 2017, Mellit.

More generally, $\varepsilon \in K(\mathcal{S})$, one define $\varepsilon[n]$ via virtual computation since each $\varepsilon \in K(\mathcal{S})$ can be written as

$$\varepsilon = [E_1] + [E_2].. + [E_r] - [V_1] - .. - [V_s]$$

where E_i, V_j are vector bundles over S. then one can also ask the same question.

For line bundle $\mathcal{L}$, define the kappa classes

$$\kappa[a,b,c] := p_*(c_1(T_{\mathcal{S}/\mathcal{M}})^a c_2(T_{\mathcal{S}/\mathcal{M}})^b \mathcal{L}^c) \in A^*(\mathcal{M}) \tag{0.2}$$

Theorem 0.1.1. *$p_*(s(\mathcal{L}[n]))$ is a universal polynomial in $\kappa[a,b,c]$.*

Computation for toric surface via localization.

Assume $S = \mathbb{C}^2$. we have natural torus action $T = \mathbb{C}^* \times \mathbb{C}^*$ on $\mathbb{C}^2$ via

$$(s,t) \times (z_1, z_2) \mapsto (sz_1, tz_2)$$

It lifts to Sn. Then the fixed locus is

$$(S^n)^T = \bigsqcup_{\sigma \vdash n} I_\sigma$$

where the ideal $I_\sigma \subset \mathbb{C}[z_1, z_2]$ corresponds to a partition σ of n, ie,

$$I_\sigma =$$

Then the AB localization formula will give

$$Z(s,t,q,w)$$

Lecture 2:

Lecture 3:

Appendix: ideas of proofs to the existence of universal polynomial

1. Complex cobordism groups $\Omega^{\mathbb{C}}$:
Definition 0.1.2.
Proposition 0.1.3. *(J.Milnor, D.Quillen)*

- $\Omega^{\mathbb{C}}$ *is is a polynomial ring with generators* $\{[\mathbb{P}^n] : n \in \mathbb{Z}_{\geq 0}\}$, *ie,*

$$\Omega^{\mathbb{C}} = \mathbb{C}[[\mathbb{P}^0], [\mathbb{P}^1], \ldots]$$

- $\Omega^{\mathbb{C}}$ *has natural graded structure by dimension.*

Proof. □

Denote $[S^{[n]}] \in \Omega$ the cobordism class of Hilbert scheme of points on smooth projective surface and set

$$H(S) := \sum_{n\geq 0} [S^{[n]}] z^n \;\in\; \Omega[[z]]$$

Then shows
Theorem 0.1.4. *$H(S)$ is independent of choice of cobordism class $[S] \in \Omega$ of S.*

A.Okounkov: For $L,\ M \in Pic(S)$, and characteristics $f : K_T(S^{[n]}) \to H_T(S^{[n]})$, define

$$< f >:= \sum_n q^n \int_{S^{[n]}} f(L^{[n]}) Euler(TS^{[n]} \otimes M^{[n]}) \in H^*_T(pt)[[q]]$$

Theorem 0.1.5. *A.Okounkov*

$$< 1 >= \sum_n q^n \int_{S^{[n]}} Euler(TS^{[n]} \otimes M^{[n]}) = \prod_{n>0} (1 - q^n)^{-\int_S c_2(TS \otimes M)}$$

Define

$$< f >' := \frac{< f >}{< 1 >}$$

My comments:

1. Is there deep relation between counting theory of S and $S[n]$?

0.1.2 Theory of Tautological integral

0.1.3 Introductory talk 1: Algebraic stacks

Definition 0.1.6. *$p : \mathcal{F} \to C$ is called category fibered over C (CFG) if*
Definition 0.1.7. *$p : \mathcal{F} \to C$ is called category fibered over C if*
Definition 0.1.8. *C := category of S-scheme. one call a functor $\mathcal{M} : C \to Sets$ is a **moduli functor** if $\mathcal{M}(T)$ is a T-family of geometric objects modulo some equivalence (eg, isomorphic).*

*$\mathcal{M}$ is called **representable (fine moduli space)** if there is a S-scheme $M \in C$ st:*

$$Hom(, M) \cong \mathcal{M}$$

$\mathcal{M}$ is said to have coase moduli space if there is a S-scheme $M \in C$ and a natural transformation from $Hom(, M)$ to $\mathcal{M}$ and $1-1$ correspondence

$$Hom(S, M) \to \mathcal{M}(S)$$

usually, a moduli functor from AG is not a representable (as far as I know, the only representable one is Quot scheme or its variant). The main obstruction is the existence of isotrivial but not trivial families, ie,

$$\begin{array}{ccc} B' \times \mathcal{X}_0 = \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ B' & \xrightarrow[etale]{} & B \end{array} \tag{0.3}$$

one typical example is functor

$$\mathcal{M} : C \to Sets$$
$$T \mapsto \{iso\ classs\ of\ line\ bundles\ on\ T\}$$

Proof. □

Lemma 0.1.9. *(Yoneda lemma)*
$\mathcal{F} \to C$ is a groupoid

Proof. For $char(k) = 0$, □

- Algebraic stacks.
- DM stacks.
- Artin stacks.

Examples:

We always use $C := sch(\mathbb{C})$.

1. The moduli stack of genus g with n-pointed curve $\mathcal{M}_{g,n}$ is a separated smooth proper DM stack:

2. The moduli stack $Bund_{r,d}(C)$ of vector bundles of rank r and degree d over a smooth projective curve C over k is a stack.

3. Hilbert scheme is a fine moduli space. consider

0.1.4 Introductory talk 2: Hilbert schemes of points on surface

X :=smooth surface. define

$$X^{[n]} := \{Z \subset X : 0 - dim,\ length(O_Z) = n\}$$

Theorem 0.1.10. *$X^{[n]}$ is a smooth of* dim $= 2n$*. In particular, if X is irreducible projective, then so is $X^{[n]}$.*

Proof. □

one can also switch viewpoint and treat $X^{[n]}$ as moduli space of certain sheaves on X. eg,

$$K3^{[n]} \cong M_H(v),\ v = (1, 0, n)$$

Cohomology of $X^{[n]}$: Representation theory and geometry !

We can view $X^{[n]}$ is a resolution of quotient singularities via Hilbert-chow morphism

$$X^{[n]} \to X^{(n)}$$

in particular, $n = 2$, $X^{[2]} = Bl_{\triangle}(X^{(2)})$.
Theorem 0.1.11. *(L.Gotte)*

1. *The Pincare polynomial for $X^{[n]}$ is given by*

$$\sum_{n=0} P_z(X^{[n]})t^n = \prod_m \tag{0.4}$$

2. *The motive of $X^{[n]}$*

Proof. □

Curve case: Macdonald's formula and its generation by Maulik and Yun

Theorem 0.1.12. *C is a projective integral curve over k of arithmetic genus g with planar singularities only.*

Proof. skecth: □

Remarks:

1. For $X = K3$, the formula implies

Define incidence Hilbert scheme $X^{[m,n]} := \{(\xi, \eta) \in X^{[m]} \times X^{[n]} :\ \xi \subset \eta\}$, it's a correspondence between $X^{[m]}$ and $X^{[n]}$. naturally there is

$$\begin{array}{ccc} X^{[m,n]} & \xrightarrow{p_2} & X^{[n]} \\ {\scriptstyle p_1}\big\downarrow & & \\ X^{[m]} & & \end{array}$$

Given a homology class $\alpha \in H_k(X)$, define

$$X_\alpha^{[m,n]} := \{(\xi, \eta) \in X^{[m]} \times X^{[n]} :\ \xi \subset \eta,\ supp(\frac{I_\xi}{I_\eta}) = point \in \alpha\ \}$$

then $X_\alpha^{[m,n]} \subset X^{[m]} \times X^{[n]}$ is an irreducible closed subscheme of $\dim X_\alpha^{[m,n]} = 2m + (n-m)k$. In particular, if α is a curve in S, then $X_\alpha^{[m,n]} \subset X^{[m]} \times X^{[n]}$ is a Langarigian submanifold.

Fork space

$$\mathbb{F} := \bigoplus_{n \geq 0} H^*(X^{[n]}, \mathbb{Q})$$

H.Nakajima defined creation and annihilation operation on $\mathbb{F}$ for each $\alpha \in H_*(X)$:

$$i > 0,\quad a_i(\alpha) : \mathbb{F} \longrightarrow \mathbb{F}\ \ by\ \ \theta \mapsto (p_2)_*(p_1^*\theta \cap X_\alpha^{[m,m+i]})$$

for each $\theta \in H^*(X^{[m]})$.

Theorem 0.1.13. *(Nakajima, Grojnouski)*

For $m, n \in \mathbb{Z} - \{0\}$ and $(-1)^{deg(\alpha)\cdot deg(\beta)} = 1$, then

$$[a_m(\alpha), a_n(\beta)] = (-1)^{m-n} m < \alpha, \beta > \delta_{m,n} id \tag{0.5}$$

where

Proof. □

Theorem 0.1.14.

Proof. □

0.1.5 Introductory talk 3: Algebraic cycles

Theorem 0.1.15.

Proof. □

Theorem 0.1.16.

Proof. □

0.1.6 Introductory talk 4: VHS

Theorem 0.1.17.

Proof. □

Theorem 0.1.18.

Proof. □

0.1.7 Talk 1: Perservese sheaf, Hilbert scheme and $P = W$ conjecture

Talk given by Junliang Shen **1. Motivation:** $P = W$ **conjecture**

Assume C smooth projective curve over $\mathbb{C}$ and $G = GL(n)$. Fixed rank=r and degree d st: $(r, d) = 1$.

$M^H := \{(E, \theta)) : s.s\ Higgis\ bundle\}$

$M^C := Hom(\pi_1(C), G)//G$

The character variety is an affine variety of dim = g. The remarkable theorems of nonabelian Hodge theory shows

Theorem 0.1.19. *(S.Donaldson, Yau-Ulenbeck, C.Simpons,..)*
There is real analytic isomorphism

$$M^H \cong M^C$$

Naturally, it implies

$$H^*(M^H, \mathbb{Q}) \cong H^*(M^C, \mathbb{Q}) \tag{0.6}$$

But How about the Hodge structure ?

The so-called $P = W$ conjecture of MAA De Cataldo, T.Hausel, L.Migliorini (see **??**) predicts

$$P_k \cong W \tag{0.7}$$

In the same paper, then should the case $rank = 2$.

2. Perseverse filtration

0.1.8 Talk 2: Application of Mixed spin fields

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Physics side: BCOV's holomorphic anomaly equation 93 (see **??**), Yamachi-Yau's polynomial structure 04 (see **??**),

Mathematical side:

- Givatal, LLY, 96, $g = 0$.
- A.Zinger, Li, Vakil, 05, $g = 1$.

Qunitic CY and construction of its mirror:

$$X_\psi = \{z_1^5 + ..z_1^5 + \psi z_1 \cdot \ldots \cdot z_5 = 0\}$$

The 1 dimesional family mirrors $\widehat{Q}_\psi$ of $Q = \{z_1^5 + ..z_1^5 = 0\}$ is resolution of X_ψ/G.

Put

$$I_0 + hI_1 + h^2 I_2 + h^3 I_3 = e^{ht} \tag{0.8}$$

consider $\mathbb{C}^5 \times \mathbb{C}$ with coordinate $(x_1, .., x_5, p)$ and $\mathbb{C}^*$ action of weight $(1, 1, 1, 1, 1, -5)$. Then two GIT quotients can be identified as

$$(\mathbb{C}^5 - 0 \times \mathbb{C})/\mathbb{C}^* = K_{\mathbb{P}^4}$$

$$(\mathbb{C}^5 \times \mathbb{C}) - 0/\mathbb{C}^* = [\mathbb{C}^5/\mathbb{Z}_5]$$

0.1.9 Talk 3: Cosection localization and quantum singularity theory

.
In 1993, E.Witten (see 0.50) propose to count

$$\#\{Spin \ \ curves \ \ + \ sections \ + Witten's \ equation\}$$

LG/CY correspondence :

$$GW(Q) \Leftrightarrow FJRW(\mathbb{C}^5/\mu_5 \xrightarrow{W} \mathbb{C})$$

History:

- FJR (2013): math theory for Witten's idea via analytical methods (cohFT) (see 0.50).
- Po - (2016) algebraic methods via matrix factorization and Hoshild homology (see 0.50).
- Chang-Li-Kiem (2015) algebraic methods via cosection localization (see 0.50) which works for narrow sectors only.

Recently, the work of JunLi and Kiem (see 0.50) extends theory so that it works for all sectors !

Borel-More homology For a topological space X, which can be embedded into a m-dimensional smooth manifold M as a closed subset, then one defines

$$H_i^{BM}(X) := H^{m-i}(M, M - X) \tag{0.9}$$

where the relative cohomology is the usual singular cohomology.

Fact: the definition is independent of the choice of embedding.

Borel-More homology has the following basic properties:

1. Proper pushforward
2. Flat pullback
3.
4. If X is compact and locally contractible, then

$$H_i^{BM}(X) \cong H_i(X)$$

by Alexander-Lefschetz.

Theorem 0.1.20. *(Kiem-Li, see* **??***)*
$\mathcal{M}$ DM stack with obstruction theory. If $U \subset \mathcal{M}$ open st: $\sigma : obs|_U \twoheadrightarrow \mathcal{O}_U$, ie, $\sigma \in H^0(obs^\vee|_U)$ is a cosection. Then $\exists$ virtual cycle $[\mathcal{M}]^{vir}_{loc} \in A_*(\mathcal{M}(\sigma))$ st:

$$\iota_*[\mathcal{M}]^{vir}_{loc} = [\mathcal{M}]^{vir} \in A_*(\mathcal{M})$$

where $\iota : \mathcal{M}(\sigma) := \mathcal{M} - U \hookrightarrow \mathcal{M}$ open inclusion.

Proof. $E \longrightarrow \mathcal{M}$ vector bundle with cosection $\sigma : E|_U \longrightarrow \mathcal{O}_U$. $E(\sigma) := E|_U \cup \ker(\sigma)$

$$\begin{array}{ccc} E(\sigma) & \xrightarrow{\tilde{\iota}} & E \\ \downarrow & & \downarrow \\ \mathcal{M}(\sigma) & \xrightarrow[\iota]{} & \mathcal{M} \end{array}$$

The localised virtual cycle $[\mathcal{M}]^{vir}_{loc} := 0^!\mathfrak{C}_\sigma$ is given by localised Gysin map where $\mathfrak{C}_\sigma :=$ is the intrinc normal cone □

0.1.10 Talk 4: Debarre-Voisin variety

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$V :=$ 10 dimensional $\mathbb{C}$ vector space. $0 \neq \sigma \in \bigwedge^3 V$, define

$$K_\sigma := \{U \in Gr(6, V) : \ \sigma|_{\bigwedge^3 U} = 0\} \tag{0.10}$$

Theorem 0.1.21. *(Debarre-Voisin, see 0.50)*

- *For generic $\sigma \in \bigwedge^3 V$, K_σ is a HK of type $K3^{[2]}$.*
- *$q_{BB}(P_\sigma) = 22$ for the Plucker polarization P_σ for K_σ.*
-

Proof. □

consider GIT moduli space

$$\mathfrak{m}_{DV} := \mathbb{P} \bigwedge^3 V // PGL(V) \supset \mathfrak{m}^o_{DV}$$

where $\mathfrak{m}^o_{DV}$ is the moduli space of smooth DV varieties.

Let $\Lambda = E_8(-1)^2 \oplus U^3 \oplus A_1$ be the lattice of $K3^{[2]}$ and $v \in \Lambda$ with $v^2 = 22$. $\Lambda_v := v^\perp$. the period domain

$$\mathcal{D}/\Gamma, \quad \mathcal{D} := \{z \in \mathbb{P}\Lambda_{\mathbb{C}} : \ (z, z) = 0, \ (z, \bar{z}) > 0 \ \}^+$$

where Γ is monodromy group. Markanman have computed the monodromy group for $K3^{[n]}$:

The period map

$$\rho_{DV} : \mathfrak{m}_{DV} \longrightarrow (\mathcal{D}/\Gamma)^{BB}$$

Question: what's the image ρ_{DV} ? and boundary ?

0.1.11 Talk 5: Bott vanishing

Theorem 0.1.22. *(R.Bott see 0.50) $L \in Pic(\mathbb{P}^n_{\mathbb{C}})$ ample $X := \mathbb{P}^n$, then*

$$H^i(X, \Omega^j_X \otimes L) = 0, \ \forall \ i, j > 0 \tag{0.11}$$

Proof. □

In general, we call X has Bott vanishing (BV) if 0.50 holds.

Remarks:

1. $j = \dim X$, it is Kodaria vanishing.
2. More generally, $E \to X$ is positive holomorphic bundle on a n-dimensional compact complex manifold, then Akizuki-Kodaira-Nakano vanishing theorem states

$$H^i(X, \Omega_X^j \otimes E) = 0, \ \forall \ i + j > n + 1 \tag{0.12}$$

3. X is any fano
4. X= n dimensional smooth fano projective st: $\exists$ perfect pairs

$$\Omega_X^1 \times \Omega_X^{n-1} \to \Omega_X^n \cong \omega_X$$

then $TX \cong \Omega_X^{n-1} \otimes \omega_X^\vee$, thus

$$H^1(X, TX) = H^1(X, \Omega_X^{n-1} \otimes \omega_X^\vee) = 0$$

by BV. This implies X is rigid, ie, no deformation of complex structures !

Corollary 0.1.23.

BV holds only for finitely many smooth projective n dimensional fano varieties for all n.

Proof. The discussion in remarks shows it has only 0-dimensional moduli. □

Theorem 0.1.24. *(V.I.Danilov) all smooth toric projective varieties satisfy BV over any field* k.

Proof. **sketch** □

eg: $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a))$ Hirebruch surface. It is toric only for a and BV holds.

Bott vanishing for $K3$:

Let X be a K3 surface and L ample. Then BV is equivalent to

$$H^1(\Omega^1_X \otimes L) = 0$$

Lemma 0.1.25. $\chi(\Omega^1_X \otimes L) = h^0(\Omega^1_X \otimes L) - h^1(\Omega^1_X \otimes L) = L^2 - 20$

Proof. It is a consequence of vanishing and HRR.

Note that

$$c_1(\Omega^1_X \otimes L) = c_1(\Omega^1_X) + 2c_1(L)$$
$$c_2(\Omega^1_X \otimes L) = c_2(\Omega^1_X) + c_1(\Omega^1_X)c_1(L) + c_1(L)^2$$

□

Remark: BV fails for $L^2 < 20$.

Lemma 0.1.26. *(Burt.Totaro)*
BV is a zariski open property.

Proof. □

Examples:

1. $X = \{f = 0\} \in |\mathcal{O}_{\mathbb{P}^3}(4)|$ smooth quartic K3, then $Pic(X) = \mathbb{Z}H$, $H = \mathcal{O}_X(1)$, $H^2 = 4$ïijŇ $L := aH$ for $a \geq 3$ so that $L^2 = 4a^2 > 20$. For any HK, a global nonzero holomorphic 2-form σ will induce $TX \cong \Omega^1_X$ so

$$H^1(X, \Omega^1(a)) = H^1(X, TX(a))$$

The Kodiara-Nadel vanishing implies

$$H^2(X, \Omega^1(a)) = H^2(X, TX(a)) = 0$$

The standard sequence

$$0 \to TX \to T\mathbb{P}^3|_X \to N_X = \mathcal{O}_X(4) \to 0 \tag{0.13}$$

It gives long exact sequence of cohomology

$$\begin{aligned} 0 \to H^0(TX(a)) \to H^0(T\mathbb{P}^3(a)|_X) \xrightarrow{\mu} H^0(\mathcal{O}_X(4+a)) \to \\ H^1(TX(a)) \to H^1(T\mathbb{P}^3(a)|_X) \to H^1(\mathcal{O}_X(4+a)) = 0 \end{aligned} \tag{0.14}$$

Note that if μ is surjective, then

$$h^1(TX(a)) = h^0(TX(a)) - \chi(TX(a))$$
$$= h^0(T\mathbb{P}^3(a)|_X) - h^0(\mathcal{O}_X(4+a)) - 4a^2 + 20 = 0$$

combining the Euler sequence

$$0 \to \mathcal{O}_X \to \bigoplus_4 \mathcal{O}_X(1) \to T\mathbb{P}^3|_X \to 0 \tag{0.15}$$

one can identify

$$\begin{aligned} H^0(T\mathbb{P}^3(a)|_X) = \{\overrightarrow{v} = (v_0, v_1, v_2, v_3) :\ v_i \in H^0(\mathcal{O}_{\mathbb{P}^3}(a+1))\} \\ \equiv (f)\ \&\ v_0 = v_1 = v_2 = v_3 \end{aligned} \tag{0.16}$$

and

$$H^0(N_X(a)) = \{w \in H^0(\mathcal{O}_{\mathbb{P}^3}(a+4))\} \equiv (f) \tag{0.17}$$

the restriction mapμ to the normal direction is given by

$$\overrightarrow{v} \mapsto \overrightarrow{v} \cdot \nabla f = \sum v_i \frac{\partial f}{\partial x_i}$$

Observe that there is a decomposition (actually it comes from exact sequence 0.50)

$$\mathbb{C}[x_0, .., x_3]_{a+4} = f \cdot \mathbb{C}[x_0, .., x_3]_a \oplus H^0(N(a))$$

so we can identify

$$Im(\mu) = \{\sum v_i \frac{\partial f}{\partial x_i} :\ v_i \in H^0(N(a+1))\}$$

If $f = x_0^4 + \ldots x_3^4$, ie, X is Fermat K3, then $\frac{\partial f}{\partial x_i} = 3x_i^3$, for $a = 3$, then

$$x_0^2 x_1^2 x_2^2 x_3 \in H^0(N(3)) = H^0(\mathcal{O}(7),\ x_0^2 x_1^2 x_2^2 x_3 \notin Im(\mu)$$

So Bott-vanishing fails for Fermat K3 w.r.t $L = \mathcal{O}(3)$.

2. $X \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(2,3)|$, then $Pic(X) = \mathbb{Z}H_1 \oplus \mathbb{Z}H_2$ and

$$H_1^2 =,\ H_1.H_2 =,\ H_2^2 =$$

By Kleiman criterion, The ample cone of X

$$Amp(X) = \{L = aH_1 + bH_2 :\}$$

consider the bott vanishing locus in moduli space of q-p K3

$$K_g^{BV} := \{(S, L) \in K_g : \ H^1(TX \otimes L) = 0\}$$

or

$$K_g^{BV,n} := \{(S, L) \in K_g : \ H^1(TX \otimes L^n) = 0\}$$

The following basic questions will be interesting

- when K_g^{BV} and $K_g^{BV,n}$ is nonempty ? Is there numerical criterion and geometric explanation ?
- If these locus nonempty, the $[K_g^{BV}] \in H^*(K_g)$ tautological ?

3. For a $(X, L) \in K_g$ with L very ample, $\varphi : X \longrightarrow \mathbb{P}^g$, one has

$$\begin{aligned} 0 \to H^0(TX \otimes L) \to H^0(T\mathbb{P}^g|_X \otimes L) \xrightarrow{\mu} H^0(N \otimes L) \to \\ H^1(TX \otimes L) \to H^1(T\mathbb{P}^g|_X \otimes L) \to H^1(\mathcal{O}_X(N \otimes L) = 0 \end{aligned} \tag{0.18}$$

4. X is a Calabi-Yau n-fold

Key observation:

Consider two groups of moduli stack $\mathcal{F}_g$, $\mathcal{K}_g$ and $\mathcal{P}_g$, $\mathcal{M}_g$ defined

$$\mathcal{F}_g(B) := \{(\mathcal{V}, \mathcal{S}) \to B : K^3_{\mathcal{V}_b} = 2g-2, \ \mathcal{S} \in |-K_{\mathcal{V}_b}|\}$$

$$\mathcal{P}_g(B) := \{(\mathcal{S}, \mathcal{C}) \to B : C_b^2 = 2g-2\}$$

with natural forgetful maps

$$s_g : \mathcal{F}_g \to \mathcal{K}_g, \quad p_g : \mathcal{K}_g \to \mathcal{M}_g$$

More generally, one can pose some requirements of Picard lattice of parameterizing objects and consider $\mathcal{F}_g^R$,$\mathcal{K}_g^R$

Theorem 0.1.27. *(Mukai, Beaville)*

1. *$\mathcal{F}_g$ is a smooth*
2.

Proof. Deformation theory discussion.

Fact: For smooth pair $Y \subset X$, its deformation is controlled by the sheaf $TX(Y)$ of holomorphic vector fields on X tangent to Y:

$$obs = H^2(TX(Y)), \quad T_1 = H^1(TX(Y))$$

and the restriction map

$$0 \to TX \otimes \mathcal{I}_Y \to TX(Y) \xrightarrow{r} TY$$

induces maps of 1st order deformations of two moduli problems.

For a general $(V, S) \in \mathcal{F}_g$,

$$obs = H^2(V, TV(S)) = 0$$

For a general $(S, C) \in \mathcal{P}_g$,

$$0 \to TS\otimes \to TS(C) \to T_C \tag{0.19}$$

where $TS(C)$ is sheaf of holomorphic vector fields on S tangent to C. one see

$$obs = H^2(S, TS \otimes \mathcal{O}(C)) = 0, \quad tan = H^1(S, TS \otimes \mathcal{O}(C))$$

So the 1st deformation for $\mathcal{P}_g$ is unobstructed and it is a smooth stack. The tangent map induced by p_g at (S, C) is identified with natural cohomology maps

$$dp_g : T_{(S,C)}\mathcal{P}_g = H^1(C, TS|_C) \to T_C\mathcal{M}_g = H^1(C, N_{C/S}) = H^1(C, \Omega^1_C)$$

induced by exact sequence 0.19. thus, its kernel

$$\ker dp_g =$$

□

0.1.12 Talk 6: cubic 4-folds and noncommutative K3

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C := moduli space of smooth cubic 4-fold.

Kuznetsov component of $X \in C$ is defined as

$$Ku(X) :=< O_X, O_X(1), O_X(2) >^{\perp}$$

$$= \{E \in D^b(X) : Ext^j(O_X(i), E) = 0, i = 0, i, 2\}$$

Theorem 0.1.28. *$stab(Ku(X)) \neq \phi$. More precisely, $\exists$ connected component*

$$stab(Ku(X)^o) \xrightarrow{\varphi} P_o^+(Ku(X))$$

Proof. □

Theorem 0.1.29. *$0 \neq v \in H_{alg}$ primitive and $\sigma \in stab^o$. Then $M_\sigma(v) \neq \phi$ iff $< v, v >\geq -2$. For generic σ, $M_\sigma(v)$ is a smooth IHS of* dim $=< v, v > +2$ *and $\exists$ $l_\sigma \in N^1(M_\sigma(v))$ ample.*

Proof. □

Examples

1. $v^2 = -2$ iff $M_\sigma(v)$ is points

0.1.13 Talk 7: compactification of moduli spaces

Theorem 0.1.30. *There is a compact moduli space for KSBA stable Varieties.*

Proof. □

Lemma 0.1.31. *Log K3 with*

Proof. □

0.1.14 Talk 8: Mathematical Moonshine and curve counting

.

1.

Theorem 0.1.32.

Proof. □

Theorem 0.1.33.

Proof. □

0.1.15 Talk 9: Higgs bundle and hyperbolicities

Theorem 0.1.34.

Proof. □

0.1.16 Talk 10: fundamental groups of degenerate varieties

.

1. Motivation

Let $f : (X, \mathcal{X}) \to (Y, \mathcal{Y})$ be a morphism of log scheme $/\mathbb{C}$.
Theorem 0.1.35.

Proof. □

2. Log geometry
Definition 0.1.36. *$X \in sch(k)$ and a homomorphism of sheaves of monoids $\alpha : \mathcal{M} \to \mathcal{O}_X$ is called log structure if*

$$\alpha : \alpha^{-1}(\mathcal{O}_X^*) \xrightarrow{\cong} \mathcal{O}_X^*$$

where monoidal structure for $\mathcal{O}_X$ is given by multiplication.

eg:

X smooth variety $/k$ and $D \subset X$ normal crossing divisor. $\mathcal{M} :=$

Theorem 0.1.37.

Proof. □

0.2 Topic 2 : Construction and compactification of Moduli space

0.2.1 Prof.E.J.N. Looijenga's Distinguished lectures

Compactification of certain locally symmetric varieties related to Algebraic Geometry

$V :=$ $\mathbb{R}$-vector space of $\dim V = 2 + n$ with a bilinear form

$$<,>: V \times V \to \mathbb{R}$$

of signature $(2, n)$.

$O(V) := \{\}$

$Gr^+(2, V) := \{\alpha \in Gr(2, V) : \ \alpha > 0\}$

Baily-Borel compactification of LSS

Baily-Borel compactification of arrangements of LSS

Toric compactification LSS

Basic Example 1: $\mathbb{H} = SL_2(\mathbb{R})/SO_2(\mathbb{R})$

step1:

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{j(\tau)=} & \mathbb{C} \\ {\scriptstyle p(\tau)=e^{2\pi\sqrt{-1}\tau}} \downarrow & \nearrow_{\mu} & \\ \mathbb{D}^* & & \end{array}$$

step2: $\mathbb{H}_c := \{z \in \mathbb{H} : \ im(z) > c\}, \ \Gamma_0 :=< T >\leq SL_2(\mathbb{Z})$.

Fact: For $c \gg 0$, $z_1, z_2 \in \mathbb{H}_c$ are Γ_0-equivalent iff $z_1, z_2 \in \mathbb{H}$ is $SL_2(\mathbb{Z})$-equivalent.

This shows

$$\begin{array}{ccl} & & \mathbb{C} \\ & \nearrow^{\mu} & \\ \mathbb{D}_c^* := p(\mathbb{H}_c) = \{z \in \mathbb{C} : 0 < |z| < e^{-2\pi c}\} & & \\ & \searrow & \\ & & \mathbb{D}_c = \overline{\mathbb{D}_c^*} \end{array}$$

Then one glue $\mathbb{D}_c$ and $\mathbb{C}$ along $\mathbb{D}_c^*$ to obtain

$$\mathbb{P}^1 \cong \mathbb{C} \cup_{\mathbb{D}_c^*} \mathbb{D}_c$$

Basic Example 2: Siegal upper half planes $\mathbb{H}_g$

0.2.2 Prof.Radu Laza's Distinguished lectures

Birational geometry of the moduli of K3 surfaces, and applications

Eg: $E := \mathbb{C}/\Lambda_\tau,\ \Lambda_\tau := \mathbb{Z} + \tau\mathbb{Z},\ \tau \in \mathbb{H}$

Classical theory shows $E \hookrightarrow \mathbb{P}^2$ holomorphically embedded by Weistrass function

$$z \mapsto [1, \varrho(z), \varrho'(z)], \quad (\varrho'(z))^2 =$$

where $\varrho(z) := \sum\limits_{\lambda \in \Lambda_\tau - \{0\}} \frac{1}{(z-\lambda)^2}$. ie, each elliptic curve can be realised as a cubic plane curve.

Theorem 0.2.1. *There is isomorphism between two compactification*

$$(\mathbb{H})^{BB}/SL_2(\mathbb{Z} \cong |\mathcal{O}_{\mathbb{P}^2}(3)|//SL(3)$$

with boundary correspondence $\infty \Leftrightarrow$ strictly semistable cubics. Here strictly semistable cubics consist of

- *3 lines $xyz = 0$*
- *line with conic*
- *Nodal cubic.*

Proof. □

In general, there are 3 ways to construct moduli space in AG:

- GIT

  It Needs to fix an embedding but unfortunately it has no canonical choice of the embedding and usually the asymtopic GIT fails.

- Hodge theory

  It is excellent when it works and translate problems into lattice. But it depends heavily on Torelli theorem.

- Moduli approach: KSBA

  It depends on the techniques from Birational geometry but hard to explicitly describe.

0.2.3 Prof.Nagaiming Mok's Distinguished lectures

uniformations of locally symmetric space

Differential geometry of locally symmetric spaces:

$\Omega \subset \mathbb{C}^n$ bounded domain with $\Gamma \subset Aut(\Omega)$ discrete subgroup.

$X := \Omega/\Gamma$

some rigidity properties of locally symmetric spaces

Bergman metric

$$\mathcal{H}^2(\Omega) := \overline{\{\varphi \in C^\infty(\Omega) :\| \varphi \|_2 := \int_\Omega d\mu(|\nabla\varphi|^2 + |\varphi|^2) < \infty\}}$$

choosing an orthornormal basis $\{f_n(z)\}$, one define the Bergman kernel as

$$K_\Omega(z, w) := \sum_n f_n(z)\overline{f_n(w)} \tag{0.20}$$

Examples:

1. $\Omega := B^n := \{z = (z_1, .., z_n) \in \mathbb{C}^n : |z|^2 \leq 1\}$ unit ball, then

2. $\Omega := B^n := \Delta^n$ polydisc, then

Background source from AG:

1. Moduli space of PPAV $/\mathbb{C}$.

By Torelli theorem of g-dimensional (A, Θ), the coarse moduli space $\mathcal{A}_g$ can be realised as the quotient of Siegel upper half plane

$$\mathcal{A}_g := \mathcal{H}_g/Sp(g),\ \mathcal{H}_g := \{M : Im(M) > 0\}$$

Q: For me, I am wondering the differential geometry will be helpful for us to understand the cohomology of locally symmetric space ?

0.3 Topic 3 : Enumerative geometry based on Moduli space

0.3.1 Prof.Yukinobu Toda's Distinguished lectures

Birational geometry for d-critical loci and wall-crossing in Calabi-Yau 3-folds

1. Motivation

$X \in sm.proj.Var(\mathbb{C})$ and $v \in \Gamma := Im(ch : K(X) \to H^{2*}(X, \mathbb{Q}))$ primitive.

For $\sigma \in Stab(X)$, $M_\sigma(v)$:= moduli space of σ-s.s. objects E in $D^b(X)$ with $ch(E) = v$.

Wall-Crossing:

Basic Questions:

1. How $M_\sigma(v)$ changes when σ crosses the wall in $Stab(X)$?
2. How $D^b(M_\sigma(v))$ changes when σ crosses the wall in $Stab(X)$?

X is K3, then

- $M_\sigma(v)$ has symplectic structure.
- $M_{\sigma^+}(v) \dashrightarrow M_{\sigma^-}(v)$ is symplectic flop (B-M).
- $D^b(M_{\sigma^+}(v)) \cong D^b(M_{\sigma^-}(v))$.

X CY 3-fold

- $M_\sigma(v)$ has (-1)-shifted symplectic structure.
-
-

$\mathfrak{m}$:= moduli stack of $E \in D^b(X)$ with $Ext^i(E, E) = 0$.

$\mathfrak{m}_\sigma(v)$:= moduli stack of σ s.s objects $E \in D^b(X)$ with $Ext^i(E, E) = 0$.

2. D-critical locus

Joyce's work shows for M is $\mathbb{C}$-scheme, $\exists$ sheaf of $\mathbb{C}$-vector space $\mathcal{S}$ st: for $\forall$ $U \subset M$ open and $U \subset V$ closed for a smooth scheme V, there is an exact sequence

$$0 \to \mathcal{S}|_U \to \mathcal{O}_V/\mathcal{I}^2 \xrightarrow{d_{DR}} \Omega^1_V/\mathcal{I}\Omega^1_V \tag{0.21}$$

ie, $\mathcal{S}|_U \cong \ker(\mathcal{O}_V/\mathcal{I}^2 \xrightarrow{d_{DR}} \Omega^1_V/\mathcal{I}\Omega^1_V)$.

eg:

A typical example is the local model: $f : V \to \mathbb{C}$ and $U := crit(f) = \{df = 0\}$, $\mathcal{I} := Im(TV \xrightarrow{df} \mathcal{O}_V)$.

Definition 0.3.1. *A **d-critical locus** is a pair (M, s) and $s \in \mathcal{S}$ st: $\forall x \in M$, $\exists x \in U \subset M$ and a closed embedding into smooth V with*

$$\begin{array}{ccc} U = \{df = 0\} & \longrightarrow & V \\ & & \downarrow f \\ & & \mathbb{C} \end{array}$$

and $s|_U = \widetilde{f}$

The most interesting example (at least for emumerative geometer) of such structure are the moduli of stable objects over CY 3-fold:

Theorem 0.3.2. *(D.Joyce)*
$X :=$ CY-3 fold, then the moduli stack $M^s_\sigma(v)$ admits a canonical d-critical locus structure.

Proof. □

Orientation for d-critical locus

in 0.50, D.Joyce shows

Theorem 0.3.3. *(D.Joyce)*
(M, s):= d-critical locus over k and M^{red}. Then there is a natural line bundle $K_{M,s}$ over M^{red} st:

$$K_{M,s}|_{U^{red}} \cong K_U^{\otimes 2}|_{U^{red}}$$

for locally $U = crit(f : V \to \mathbb{C})$.

Proof. □

So a square root can be defined and it is called a **Orientation** for (M, s).

Definition 0.3.4. *(Y.Toda)*
Two critical locus $M^+ M^-$ (schemes or algebraic space) with diagram

$$\begin{array}{ccccc} M^+ & & & & M^- \\ & \searrow^{\pi^+} & & \swarrow^{\pi^-} & \\ & & A & & \end{array} \tag{0.22}$$

is called d-critical flop at $p \in A$ if $\exists x \in U \subset A$ open and flop diagram st:

$$\begin{array}{ccc} (\pi^\pm)^{-1}(U) & \xrightarrow{\approx} & \{dw^\pm = 0\} \\ \downarrow & & \downarrow \\ U & \xrightarrow[\text{closed embdedding}]{} & Z \end{array}$$

Examples 1.

2. C:= a smooth projective curve of genus g and $n > 0$,

$$\begin{array}{ccc} S^{n+g-1}(C) & & S^{-n+g-1}(C) \\ & {\scriptstyle \pi^+}\searrow \quad \swarrow{\scriptstyle \pi^-} & \\ & Pic^{n+g-1}(C) & \end{array} \tag{0.23}$$

with $\pi^+(D) = \mathcal{O}(D), \;\; \pi^-(D') = \omega_C(D')$.

then

- $h^1(L) \geq 2$, d-critical flop.
- $h^1(L) = 1$, d-critical divisor contraction.
- $h^1(L) = 0$, d-critical Mori fibered space.

$$\begin{aligned} \mathfrak{m}_{\Theta_E}(\overrightarrow{m}) &:= [Ext^1(E,E)/Aut(E)] \\ & [\prod_{i \frown j} Hom(V_i, V_j)/\prod_i GL(V_i)] \end{aligned} \tag{0.24}$$

Wall-Crossing for 1**-dim stable sheaves:**

$Coh_{\leq 1}(X) := \{E \in Coh(X) : \dim supp(E) \leq 1\}$

$\Gamma_{\leq 1} := H_2(X, \mathbb{Z}) \oplus \mathbb{Z}$

$A(X)_{\mathbb{C}} := \{B + \sqrt{-1}\omega \in H^2(X, \mathbb{C}) : \; \omega \; ample\}$ space of complexied

consider

$$Z_{B,\omega} : \Gamma_{\leq 1} \longrightarrow \mathbb{C}$$

$$(\beta, n) \mapsto -n + (B + \sqrt{-1}\omega).\beta$$

Fact: $\sigma_{B,\omega} := (Z_{B,\omega}, Coh_{\leq 1}(X)) \in Stab_{\leq 1}(X) := D^b(Coh_{\leq 1}(X))$

$M_X(\beta)$:= moduli of pure 1-dim stable $E \in coh(X)$ with $\chi(E) = 1$ and $[E] = \beta$.

Theorem 0.3.5. *The wall-crossing diagram*

$$\begin{array}{ccc} M_{\sigma^+}(v) & & M_{\sigma^-}(v) \\ & {\scriptstyle \pi^+}\searrow \quad \swarrow{\scriptstyle \pi^-} & \\ & M_{\sigma}(v) & \end{array} \tag{0.25}$$

is a d-critical flop.

Proof. □

Wall-crossing for stable pairs moduli space
Lemma 0.3.6.

Proof. □

D/K conjecture

0.3.2 Talk 1: Analytical methods in complex algebraic geometry

Differential geometry of Holomorphic bundles:

$X :=$ kahler manifold of dim $= n$ with $E \to X$ holomorphic $rank = r$ vector bundle. Let

$$h : E \times E \to \mathbb{C}$$

be hermitian metric on E, ie, h is a fiberwise an hermitian inner product over each fiber $E_x \cong \mathbb{C}^r$. choose a good cover$\{U_\alpha\}$ for X (always exists for compact smooth manifold).

Singular Hermitian metric: For line bundle $L \to X$,
Definition 0.3.7. *$L \to X$ is called*

1. *ample if*
2. *nef if*
3. *big if*
4. *Psudo-effective if*

Relations to algebraic setting

Analytical proof of Nadel vanishing
Theorem 0.3.8. *Bochner formula*

$$[\sqrt{-1}\Theta, \Lambda] = \Delta_{\partial_E} - \Delta_{\overline{\partial}_E}$$

Proof. □

Lemma 0.3.9. *(L.Holmander)*
$(E, h) \to X$ is a holomorphic vector bundle over a compact kahler manifold (X, ω). Assume $A := [\sqrt{-1}\Theta, \Lambda]$ is positive on $\Omega^{p,q}(E), q \geq 1$, then for any $g \in L^2(X, \Omega^{p,q}(E))$ with $\overline{\partial}_E g = 0$, $\int_X (A^- 1g, g) < \infty$, there is a $f \in \Omega^{p,q-1}(E))$ st:

$$\overline{\partial}_E f = g, \ \| f \|^2 \leq \int_X (A^- 1g, g)$$

Proof. □

0.3.3 Talk 2:

0.4 Topic 4: Complex Geometry and Birational geometry

0.4.1 Prof.Mihnea Popa 's Distinguished lectures

Hodge ideals and applications

Goal of the lecture:

$X \in sm.Var(\mathbb{C})$ and $D \subset X$ reduced hypersurface. To associate a sequence of ideal sheaves $\{I_k(D)\}_{k\in\mathbb{Z}}$ which have to do with

- singularities of (X, D)
- Hodge theory of $X - D$.
- Local/global properties and applications.

set $\mathcal{D}_X$:= sheaves of differential operators

$$\partial_{x_1}^{a_1}...\partial_{x_n}^{a_n} := \frac{\partial^{a_1}}{\partial x_1^{a_1}} \circ ... \circ \frac{\partial^{a_n}}{\partial x_1^{a_n}}$$

for local coordinate $(x_1, ..., x_n)$.

$\mathcal{D}_X$ has natural filtration $F^\bullet \mathcal{D}_X$ by the order of the differential operators, ie,

$$F^l \mathcal{D}_X := \{\partial_x^a : |a| \leq l\}$$

Definition 0.4.1. *A $\mathcal{D}_X$-module $\mathcal{M}$ on X is a quai-coherent sheaf on X which is a module over $\mathcal{D}_X$*

Remark: equivalently, there is integrable connection on M

$$\nabla : \mathcal{M} \to \mathcal{M} \otimes \Omega_X$$

st: $\nabla^2 = 0$.

such a connection will also give representation for $\pi_1(X) \longrightarrow End(\mathcal{M}_x)$ via monodrmy

Examples;

1. $\mathcal{O}_X(*D) := \bigcup_k \mathcal{O}_X(kD)$

2. $E \longrightarrow X$ a holomorphic vector bundle.

Definition 0.4.2. *one define the filtration od ideal sheaves $I_\bullet(D)$ by*

$$R^0 f_* C^\bullet = \omega_X((k+1)D) \otimes \mathcal{D}_X \tag{0.26}$$

one can show that

a independent of choice of log resolutions.

b contains Saito's Hodge filtration.
Theorem 0.4.3.

Proof. □

Examples;

1.X surface

2.$x \in D$ ordinary singular point with $mul_x(D) = m$, ie, projective tangent cone at x is smooth , a typical example is cone vertex of $deg = m$ hypersurface in $\mathbb{P}^{n-1}$. then

- $I_0(D) = \mathfrak{m}_x^{m-n}$
- $I_k(D) = \mathcal{O}_X$ iff $k \leq \frac{n}{m} - 1$
-

Basic Questions

1. when $I_k(D) = \mathcal{O}_X$?
2. Given $x \in D$, when $I_k(D)_x \subset \mathfrak{m}_x^q$ for some $q > 0$?

Facts and some an

- $I_0(D) = \mathcal{O}(x)$ iff (X, D) is lc.
- Take a log resolution for (X, D)
$$f : Y \to X$$
with $E = (f^*D)_{red}$, then
$$I_0(D) = f_*(\mathcal{O}_Y(K_{Y/X} + E - f^*D)) \tag{0.27}$$

Definition 0.4.4. *(X, D) is k-lc if $I_k(D) = \mathcal{O}_X$, ie,*
$$I_0(D) = I_1(D) = \ldots = I_k(D) = \mathcal{O}_X$$

Theorem 0.4.5. *(M.Saito, see 0.50)*
(X, D) is k-lc iff
$$\alpha_f - 1 \geq k$$

Proof. □

Theorem 0.4.6.

Proof. ◻

$(\mathcal{M}, F)$ is filtered $\mathcal{D}_X$-module, DeRham complex

$$DR(\mathcal{M}) := \{\mathcal{M} \xrightarrow{\nabla} \mathcal{M} \otimes \Omega_X^1 \xrightarrow{\nabla} \mathcal{M} \otimes \Omega_X^2 \to \dots\}$$

$$F_k DR(\mathcal{M}) := \{F_k\mathcal{M} \xrightarrow{\nabla} F_{k+1}\mathcal{M} \otimes \Omega_X^1 \xrightarrow{\nabla} F_{k+2}\mathcal{M} \otimes \Omega_X^2 \to \dots\}$$
$$gr_k^F \mathcal{M} := F_k\mathcal{M}/F_{k-1}\mathcal{M}$$

Theorem 0.4.7. *Saito vanishing*
X Projective and $(\mathcal{M}, F)$ is filtered $\mathcal{D}_X$-module, L is ample line bundle, then

$$\mathbb{H}^i(X, gr_k^F \mathcal{M} \otimes L) = 0, \ \ \forall \ i > 0 \tag{0.28}$$

Proof. ◻

Examples;

1. $\mathcal{O}_X$ with trivial filtration

$$F_k\mathcal{O}_X = \begin{cases} \mathcal{O}_X & \text{if } k \geq 0, \\ 0 & \text{if } k < 0, \end{cases}$$

then

$$DR(\mathcal{O}_X) = \{\mathcal{O}_X \xrightarrow{\nabla} \Omega_X^1 \xrightarrow{\nabla} \Omega_X^2 \to \dots\}$$
$$gr_{-k}^F DR(\mathcal{O}_X) = \Omega_X^k[n-k]$$

In the case, Saito vanishing ⇔ Kodaira-Nakano vanishing :

$$H^p(X, \Omega_X^q \otimes L) = 0, \ p + q > n \tag{0.29}$$

for L ample.

0.4.2 Prof.Mircea Mustata's Distinguished lectures

Hodge ideals and singularities

0.4.3 Prof.Junyan Cao's Distinguished lectures

Singular hermitian metrics and some applications in complex geometry

1. Introduction: Iitaka conjecture

X smooth projective variety $/\mathbb{C}$. The Kodiara dimension of X is defined to be

$$\mathfrak{K}(X) :=$$

$\mathfrak{K}(X)$ is a basic birational invariant of X. More generally, for a pesudo-effective divisor $D \in Eff(X)$, one defines its Kodiara-Iitaka dimension

Iitaka Conjecture $C_{m,n}$ $f : X^m \to Y^n$ fibration (here, surjective and connected fiber) of two smooth projective varieties and F is general fiber, then

$$\mathfrak{K}(X) \geq \mathfrak{K}(Y) + \mathfrak{K}(F)$$

The known result so far:

- Kawamata, Kollar
- Y ia Abelian variety by Paun-Cao.
- $\dim Y \leq 2$ by Cao.

Basic tool: Positivity of $f_*(\omega_{X/Y})$

2. SHM on holomorphic bundles

$H_r := \{A = (a_{ij})_{r\times r} : \ a_{ij} \in \mathbb{C}, A = A^*, A > 0\}$ space of Hermitian matrix.

$E \xrightarrow{\pi} X$ complex vector bundle with good cover $X = \bigcup\limits_{\alpha} U_\alpha$ and with trivialization $\{\psi_\alpha\}$

$$\begin{array}{ccc} U_\alpha \times \mathbb{C}^r & \xrightarrow{\psi_\alpha} & E|_{U_\alpha} = \pi^{-1}U_\alpha \\ & \searrow & \downarrow \pi \\ & & U_\alpha \end{array}$$

A **Singular Hermitian metric** on E is a function $h : X \to \overline{H}_r$ st: $0 < det(h) < \infty$ a.e. on X

In particular, when h is C^∞, this is the usual smooth metric. when h is holomorphic, then there is unique Chern connection on E (just as Levi-Civita connection is unique in Riemannian geometry w.r.t a given Riemannian metric)

Fact: The curvature current of h on E is given by

$$\sqrt{-1}\Theta_h = \sqrt{-1}\,\overline{\partial}(h^{-1}\partial h) \in (\Omega^{1,1}(E))^* = (\Gamma(End(E) \otimes \Omega^{1,1}))^*$$

Definition 0.4.8. *(Griffith)* (E, h) *is called Griffith semi-positive (GSP) if for* $\forall\ x \in X,\ v \in T^{1,0}X, e \in E_x$,

$$(\sqrt{-1}\Theta_h(v, \overline{v})e, e),)_h > 0$$

where $(,)_h$ *is the hermitian metric induced by* h.

Remark:

1.

Definition 0.4.9. *(Viewheg)* $\varepsilon \in Coh(X)$ *is called weak positive (WP) w.r.t* $A \in Amp(X)$ *if for* $\forall\ m \in \mathbb{N},\ \exists\ k := k(m, X) \in \mathbb{N}$ *st:*

$$H^0(X, (Sym^{mk}(\varepsilon))^{\vee\vee} \otimes A^k) \twoheadrightarrow ((Sym^{mk}(\varepsilon))^{\vee\vee} \otimes A^k)_x$$

for generic $x \in X$.

Theorem 0.4.10. *(Viewheg)*

$f : X \longrightarrow Y$ *fibration of two smooth projective varieties, then* $f_*((\omega_{X/Y})^m)$ *is WP*

Proof. □

0.4.4 Prof.Dabaree's Distinguished lectures

Hyperkahler varieties

0.4.5 Prof.Junk Huang's Distinguished lectures

Minimal

1. Motivation: Deformation of Grassmanian

$Gr(p,q) := \{L \subset \mathbb{C}^{p+q} : \dim L = p\}$.

By R.Bott, $H^1(Gr(p,q), TGr(p,q)) = 0$, then $Gr(p,q)$ is locally rigid, ie, for any smooth family $\mathcal{X} \to B$ with $\mathcal{X}_b \cong Gr(p,q)$ for some $b \in B$, then $\mathcal{X}_t \cong Gr(p,q)$ for general $t \in B$.

0.4.6 Talk 1: Introduction to multiple ideal sheaves

$X \in sm.Var(\mathbb{C})$, $D \geq 0$ $\mathbb{Q}$-divisor and $\mathfrak{a} \subset O_X$, $c \in \mathbb{Q}_{>0}$. multiple ideal sheaves $J(D), J(\mathfrak{a}^c)$ measure the "singularity" of D or $\mathfrak{a}$. The pholo is D_1 more singular than D_2, then $J(D_1) \subset J(D_2)$

Analytical construction

Assume local equation for D_i is f_i and $D = \sum a_i D_i$, then

$$J(D) := \{h \in O_X : \frac{|h|^2}{\sum |f|^{2a_i}} \in L^1_{local}\} \tag{0.30}$$

$$J(\mathfrak{a}^c) := \{h \in O_X : \frac{|h|^2}{(\sum |f_i|^2)^c} \in L^1_{local} \ \forall f_1, ..., f_r \ generator \ of \ \mathfrak{a}\} \tag{0.31}$$

Algebraic construction

Take log resolution $\mu : Y \longrightarrow X$ of pairs (X, D) or $(X, \mathfrak{a})$, ie,

- Y is a smooth variety.
- μ is proper birational morphism.
- $Ex(\mu) + \mu^* D$ is snc or $Ex(\mu) + F$ is snc for

$$\mathfrak{a} O_Y = O_Y(-F)$$

By Hioranaka's resolution of singularities, the log resolution always exists. Then one defines

$$J(D) := \mu_* O_Y(K_{X/Y} - \lfloor \mu^* D \rfloor) \tag{0.32}$$

$$J(\mathfrak{a}^c) := \mu_* O_Y(K_{X/Y} - \lfloor cF \rfloor) \tag{0.33}$$

Proposition 0.4.11. *1. The definitions are independent of choice of log resolutions.*

2. $J(D)$ $(J(\mathfrak{a}^c))$ $\subset O_X$ *is idea sheaf whose*

Proof. 1. Key fact: any 2 log resolution can be dominanted by another one, ie,

$$\begin{array}{ccc} Y_3 & \longrightarrow & Y_2 \\ \downarrow & & \downarrow \\ Y_1 & \longrightarrow & X \end{array}$$

□

Invariants from multiple ideal sheaves

Definition 0.4.12. *pairs* (X, D) *is klt if* $J(D) = O_X$

lc if $J((1-\epsilon)D) = O_X$ *for* $0 < \epsilon < 1$.

Note that $J(cD)$ is trivial for $c \ll 1$ and nontrivial for $c \gg 1$. Then one can define log canonical throshed via multiple ideal sheaves

$$lct_x(D) := \inf\{c : \ J(cD)_x \subset \mathfrak{m}_x\}$$

Examples

1. $\mathfrak{a} =< x_1^{l_1}, ..., x_n^{l_n} >\subset \mathbb{C}[x_1, ...x_n]$, then at origin $0 \in \mathbb{C}^n$

$$lct_0(\mathfrak{a}) = \sum \frac{1}{l_j}$$

2.

Theorem 0.4.13. *M.Mastata 2002*

$$lct(\mathfrak{a}) = \dim X - \sup \frac{\dim Arc_m(\mathfrak{a}}{m+1}$$

Proof. □

Vanishing results

Theorem 0.4.14. *Nadeal vanishing*
Assume $D \geq 0$ $\mathbb{Q}$*-divisor, then*

1. $R^i\mu_* O_Y(K_{X/Y} - \lfloor \mu^* D \rfloor)$ *for any* $i > 0$ *and log resolution* $\mu : Y \to X$,
2. *If* X *is projective and* $L - D$ *nef and big, then*

$$H^i(X, O_X(K_X + L) \otimes J(D)) = 0, \quad i > 0$$

Proof. □

Subadditivity

Theorem 0.4.15.

$$\begin{aligned} J(D_1 + D_2) &\leq J(D_1) \cdot J(D_2) \\ J(X, \mathfrak{a}^c \cdot \mathfrak{b}^d)) &\leq J(X, \mathfrak{a}^c) \cdot J(X, \mathfrak{b}^d) \end{aligned} \tag{0.34}$$

Proof. □

Asymtopic construction

Proposition 0.4.16.

Proof. □

Mumford regularity theorem

Definition 0.4.17. *Let $B \to X$ be a ample and BPF line bundle and $\mathcal{F} \in Coh(X)$ is called $m-$reularity w.r.t to B if*

$$H^i(X, \mathcal{O}((m-i)B) \otimes \mathcal{F}) = 0, \ \forall \ i > 0$$

Proposition 0.4.18. *(D.Mumford)*

Proof. □

J.Kollar's theorem on singularities of PPAV (A, Θ)

Theorem 0.4.19. *(J.Kollar)*
Let (A, Θ) be a PPAV, then (A, Θ) is lc.

Proof. Sketch: proof by contradication via multiple ideal sheaves.

$\exists \ \epsilon \in (0,1)$ st:$J((1-\epsilon)\Theta) \subset \mathcal{O}_A$. Then set $Z := zeros(J((1-\epsilon)\Theta)) \subset \Theta$ □

Age and Siu's theorem

Theorem 0.4.20. *(J.Kollar)*
$L \to X$ ample line bundle over smooth projective bundle and for any irreducible subvariety $Z \subset X$, $L^{\dim Z}.Z$

Proof. □

Siu's theorem on invariance of plurigenra

Masuta 's big thorem

0.4.7 Talk2: Introduction to Bridgeland stability condition

.

$\mathcal{D}$:= triangled category.
Definition 0.4.21. *A t-structure for $\mathcal{D}$ is a pair $(\mathcal{D}^{\geq 1}, \mathcal{D}^{<0})$ stïÿŽ*

The heart $\mathcal{A}$ of the t-structure $(\mathcal{D}^{\geq \geq 1}, \mathcal{D}^{<0})$ is defined

$$\mathcal{A} := \mathcal{D}^{\geq 1} \cap \mathcal{D}^{<0}$$

The basic also motivated example to keep in mind is

$$\mathcal{D} := \mathcal{D}^b(X), X \in sm.proj.var(\mathbb{C})$$

$$\mathcal{D}^{\geq 1} := \{E^\bullet \in \mathcal{D}^b(X) : \ H^i(E^\bullet) = 0, \ \forall i \geq 1\}$$

$$\mathcal{D}^{<0} := \{E^\bullet \in \mathcal{D}^b(X) : \ H^i(E^\bullet) = 0, \ \forall i < 0\}$$

In this case, the heart is just $Coh(X) = \mathcal{D}^{\geq 1} \cap \mathcal{D}^{<0}$

0.4.8 Talk3: The image of period map of IHS 4-fold of $K3^{[2]}$ type

Theorem 0.4.22.

Proof. □

0.4.9 Talk 4: Dual complex of Fano variety and

.

1. Vanishing theorem in *char* > 0.

Theorem 0.4.23.

Proof. □

Definition 0.4.24.

2. Dual complex

Definition 0.4.25.

Theorem 0.4.26. *(de Fernex-Kollar-Xu)*
(X, Δ) *is lc and* $(Y, \Delta_Y) \to (X, \Delta)$ *is dlt blowup, then the dual complex* $\mathcal{D}(\Delta_Y^{=1})$ *is independent of the choice of dlt blowup up to homeomorphism.*

Proof. □

0.4.10 Talk 5: Sehardri constant

Definition 0.4.27.
Theorem 0.4.28.

Proof. □

0.4.11 Talk 6: construction of Non-kahler CY 3-fold

Problem: How many topological types of CY 3-folds ?

eg: M.Freedman construct
Theorem 0.4.29. *$\forall a \in \mathbb{Z}_{>0}$, $\exists X_a$ Non-kahler CY 3-fold with*

$$b_2(X_a) = a + 3$$

Proof. □

Deformation of SNC CY *n*-folds
Theorem 0.4.30. *(Kawamata-N)*
$X := X_i$ SNC CY n-fold st:

- $H^{n-1}(X, \mathcal{O}_X) = 0, \;\; H^{n-2}(X_i, \mathcal{O}_{X_i}) = 0.$
- *X is d-semistable, ie,*

$$(\otimes \frac{I_i}{I_i I_D})^\vee = \mathcal{O}_D$$

Then $\exists\, \phi : \mathcal{X} \to \Delta$ st:

1. *$\mathcal{X}$ is smooth.*
2. *$\mathcal{X}_t$ is CY n-fold for $t \neq 0$.*

Proof. □

0.4.12 Talk : Nef $-K_X$ and RC fibration

Definition 0.4.31.
Theorem 0.4.32. *$f : X \to Y$ of normal projective varieties with $f_*\mathcal{O}_X = \mathcal{O}_Y$ and K_Y is $\mathbb{Q}$-cartier. Assume Y has canonical singularities and $\triangle = \triangle^+ - \triangle^-$ and D is $\mathbb{Q}$-cartier st:*

- *$-(K_X + \triangle^+)$ lc over general point of Y.*
- *$-(K_X + \triangle + f^*D$ is nef.*

Then $-f^*(K_X + D) + \triangle^-$ is Pesudo-effective.

Proof. □

0.4.13 Talk: Derived invariant from Albanese map

.

Motivating problems: For $X, Y \in sm.proj.Var(\mathbb{C})$ and $D(X) \cong D(Y)$, then $h^{p,q}(X) = h^{p,q}(Y)$?

Known Results

- $h^{0,q}$ by
-

0.4.14 Talk : Monodromy and degeneration of K-trivial varieties

.

The motivation is coming from trying to understand the period map $\mathcal{M} \to \mathcal{D}/\Gamma$.

Definition 0.4.33. *Dual complex*

Theorem 0.4.34. *(Kulikov-)*

Let $\mathcal{X} \to \triangle$ be a degeneration of K3 after semi-stable reduction and $K_{\mathcal{X}/\triangle} \equiv 0$. then the central fiber $\mathcal{X}_0$ is classified

Type	$\mathcal{X}_0$	*Shape* Σ	*nilponent index* ν
I	*smooth*	*point*	*1*
II	*002*		
III	*002*	*sphere* S^2	*3*

Proof. □

Theorem 0.4.35. *(J.Shah) TFAE*

- *Monodromy is finite.*
- $\mathcal{X}_0$ *has canonical singularities.*

Proof. In dim = 2, canonical singularities ⇔ Du val (ADE, or simple) singularities. □

Natural one will ask what will happen in high dimension ?

Degeneration of IHSs

Theorem 0.4.36. *(Fujino, Hacon-Xu)*

Proof. □

Theorem 0.4.37. *(Kollar-Laza-Sacca-Voisin)*

1. Finite Monodromy.

$\mathcal{X} \to \triangle$ minimal dlt projective degeneration of IHSs, ie, $K_{\mathcal{X}} \equiv 0$, $(\mathcal{X}, \mathcal{X}_t)$ dlt $\forall\ t \in \triangle$. then TFAE

1. *Finite monodromy on H^2.*
2. *$\mathcal{X}_0$ canonical singular.*
3. *$\mathcal{X}_0$ has a component which is not uniruled.*

2. Smooth Filling

$\mathcal{X} \to \triangle$ finite monodromy degeneration, then

Proof. □

Definition 0.4.38. *$\mathcal{X}_0$ is* ***Cohomologicall insignificant singular*** *if for $\mathcal{X} \to \triangle$, the specialization map*

$$sp_k : H^k(\mathcal{X}_0) \to H^k_{lim}$$

is isomorphism on $I^{0,k}, I^{k,0}$ of MHS on H^k_{lim}.

Definition 0.4.39.

Du Bois

Definition 0.4.40.

Theorem 0.4.41. *(Steenberk, Kollar-Kovacs)*

slc ⇒ Du Bois ⇒ Cohomologicall insignificant singular

Proof. □

Degeneration of CY 3-folds

Theorem 0.4.42.

Proof. □

0.4.15 Talk: Positivity of CM line bundle

.

1. K-stability

In the talk, X is Fano $\Leftrightarrow$ klt & $-K_X$ ample.

For $q \in \mathbb{Z}_{>0}$, D is a q-basis type effective divisor if

$$D = \frac{\sum\{s_i = 0\}}{h^0(-qK_X)} \sim_{\mathbb{Q}} -K_X$$

for $H^0(x, -qK_X) = span\{s_1, ..., s_{h^0(-qK_X)}\}$.

$$Lct(X, \Gamma) := \sup\{t : \ (X, t\Gamma) \ is \ lc \ \}$$

$$\delta_q(X) := \inf\{lct(D) : \ D \in | - qK_X| \ q - basis \ \}$$

Theorem 0.4.43. *(Fujino)*

$$\delta(X) := \limsup \delta_q(X) = \lim_{q\to\infty} \delta_q(X) \tag{0.35}$$

is well-defined.

Proof. ☐

2. CM line bundle

$f : X \to T$ a Fano family, ie,

- flat with normal fiber.
- $-K_{X/T}$ is $\mathbb{Q}$-cartier and ample.

In 1990s, defined

$$\lambda_f := f_*((-K_{X/T})^{n+q})$$

Theorem 0.4.44. *(Knudson-Mumford)*

Proof. ☐

3. Moduli conjecture (Y.Odaka,Tian, Donaldson)

For $n \in \mathbb{Z}_{>0}$ $v \in \mathbb{Q}_{>0}$, one expects

1. $\exists$ Moduli stack $\mathcal{M}_{n,v}^{Kss}$
2. $\exists$ algebraic space $M_{n,v}^{K-ss}$ which is a good proper moduli space for .

3. CM bundle λ gives polarization for $M_{n,v}^{K-ss}$.

0.4.16 Talk: Openness of uniform K-stabilities

Theorem 0.4.45. *X smooth Fano manifold, then*

1. $[CDS, Tian, Berman]$,

   X is K-polystable iff X admits KE.

2. $[Berman - Bouksom - Jonsson]$,

   X is uniform K-stable iff X admits KE and $\#Aut(X) < \infty$.

Proof. □

Theorem 0.4.46. *(Blum-Liu, 2018)*
$\mathcal{X} \xrightarrow{\pi} T$ is a $\mathbb{Q}$-Fano family, then

1. $\{t \in T : \mathcal{X}_t \ uniform\ K - stable\} \subset T$ *is zariski open*
2. $\{t \in T : \mathcal{X}_t\ K - semistable\} \subset T$ *is a countable intersection of zariski open subsets.*

Proof. □

Remark:

Examples:

1. all smooth quadrics $Q^n \subset \mathbb{P}^{n+1}$ are K-polystable.

2. $\dim X = 2$, for smooth case, By Tian, the only obstruction to KE is Futaki invariant:

- K-polystable: $\mathbb{P}^2,\ \mathbb{P}^1 \times \mathbb{P}^1,\ Bl_{3\ pts}\mathbb{P}^2$.
- K-stable: $1 \le (-K_X)^2 \le 5$.
- K-unstable: $Bl_{1\ pt}\mathbb{P}^2,\ Bl_{2\ pts}\mathbb{P}^2$

3. By K.Fujita 2016, smooth hypersurface $X_{n+1} \subset \mathbb{P}^{n+1}$ is K-stable for $N \ge 3$

4. By Odaka,

By Liu-Xu, $X_3 \subset \mathbb{P}^4$ is K-stable (s.s) iff GIT stable (s.s).

Definition 0.4.47. *$D \sim_{\mathbb{Q}} -K_X$ is m-basis type divisor if*

$$D = \sum_i \frac{\{s_i = 0\}}{m \cdot N_m}$$

where $H^0(X, -mK_X) = span\{s_1, ..., s_{N_m}\}$

Theorem 0.4.48. *(Blum-Liu, 2018)*
$\mathcal{X} \xrightarrow{\pi} T$ is a $\mathbb{Q}$-Fano family, then the function

$$t \in T \mapsto \delta(\mathcal{X}_t)$$

is lower semi-continuous, ie,

Proof. □

By Blum-Jasson 2017,

$$lct(X, D) = \inf_{E\ div/X} \{\frac{A_X(E)}{ord_E(D)}\}$$

where $A_X(E)$ is log discrepency for $f : Y \to X$ and $E \subset Y$ is a prime divisor. Thus,

$$\begin{aligned} \delta_m(X) &= \inf_D \inf_{E\ div/X} \{\frac{A_X(E)}{ord_E(D)}\} \\ &= \inf_{E\ div/X} \{\frac{A_X(E)}{\sup_D\{ord_E(D)\}}\} \end{aligned} \tag{0.36}$$

Each $E\ div/X$ induces a filtration on $R_m := H^0(X, -mK_X)$:

$$\mathcal{F}_E^\lambda := \{s \in R_m :\ ord_E(s) \geq \lambda\}$$

then

$$R_m = \mathcal{F}_E^0 \supseteq \mathcal{F}_E^1 .... \mathcal{F}_E^{mT_m} = 0$$

where $T_m := \frac{1}{m} \max\{ord_E(s) :\ s \in R_m\}$.
define

$$s_m(E) := \sup_D\{ord_E(D)\} = \frac{1}{mN_m} \sum_j \dim \mathcal{F}_E^j$$

Proposition 0.4.49.

$$\lim_{m\to\infty} s_m(E) = \frac{1}{Vol(-K_X)} \int_0^\infty Vol(-K_X - tE) dt$$

Proof. □

idea of proof

$$\widehat{\delta}_m(X) := \inf_{\mathcal{F}} \{\frac{lct(X, \mathfrak{b}_\cdot(\mathcal{F}))}{s_m(\mathcal{F})}\}$$

0.4.17 Talk: Birational rigidity

Definition 0.4.50. *$f : Y \to Z$ is called a Mori fiber space (MFS) if*

1. *Y is $\mathbb{Q}$-factorial terminal singular.*
2. *$-K_Y$ is f-ample.*
3. *$\rho(Y/Z) = 1$.*

eg: A $\mathbb{P}^r$ bundle.

$X :=$ Fano with $\rho(X) = 1$
Definition 0.4.51. *X is called Birational superrigidity (BSR) if X/pt is MFS and for any*

Q: Is there a nice moduli for BSR/BR Fanos ?

For a family of Fano varieties $\mathcal{X} \xrightarrow{\pi} T$, Is

$$BSR(\mathcal{X} \xrightarrow{\pi} T) := \{t \in T : \ \mathcal{X}_t \ BSR\}$$

constructible ? open ?
Theorem 0.4.52. *(Stibitz-zhang)*
For two families of Fano varieties

$$\begin{array}{ccc} X & & Y \\ & \searrow \quad \swarrow & \\ & C & \end{array}$$

st: $X - X_0 \cong Y - Y_0$ and $X_0, \ Y_0$ BSR, then $X \cong Y$

Counterexample for openness of BSR:

$$\mathcal{X} := \{y^2 = f(x_0, .., x_{n+1}), \ ty = g(x_0, .., x_{n+1})\} \subset \mathbb{P}(1^{n+2}, m) \times \mathbb{C}$$

For $t \neq 0$,

$$\mathcal{X}_t = \{g^2 - t^2 f = 0 \ \} \subset \mathbb{P}^{n+1}$$

0.5 Topic 5: Geometric Langlands Program

0.5.1 Prof.Zhiwei Yun's Distinguished lectures

An Introduction to the moduli of Shtukas

1. Definition of Shukas and examples

Story over number fields $F/\mathbb{Q}$ comes from R.Langlands:

To establish the connection, People

X := smooth projective geometrically connected curve $/k := \mathbb{F}_q$.

$F := k(X)$

Definition 0.5.1. *$S \in sch(k)$, $D \subset X \times S$ relative divisor. A shtukas over S with legs in D consists of pair (ε, ρ) satisfying*

1. *$\varepsilon \longrightarrow X \times S$ is a vector bundle.*
2. *$\rho : \varepsilon|_{X\times S-D} \xrightarrow{\simeq} {}^{\tau}\varepsilon := (id \times Fr_S)^{*}\varepsilon|_{X\times S-D}$*

one can consider the category of Shtukas $Sht(S, D)$:

Theorem 0.5.2. *Assume $D = \phi$ and S connected. Then for any $(\varepsilon, \rho) \in Sht(S, D)$, there is an Etale covering $f : S' \longrightarrow S$ st:*

$$f^{*}() \tag{0.37}$$

Proof. □

Baby version of theorem:

Lemma 0.5.3.

Proof. □

there is a natural 1-1 correspondence

$$\frac{\{P \longrightarrow C : \ G - bundle\}}{\simeq} \leftrightarrow [G(K) \setminus G(\mathbb{A})/G(O)]$$

Given a principle G-bundle $P \longrightarrow C$,

0.6 Topic 6: Rationality Problems

0.6.1 Prof.Burt Totaro 's Distinguished lectures

Algebraic cycles and birational geometry

$X \in Var(k)$ is called rational if .

eg:

1. $\{x^2 + y^2 = 1\} \subset \mathbb{A}^2_k$ is rational for $k = \mathbb{C}, \mathbb{R}$. The birational map is given by

$$\mathbb{A}^1 \longrightarrow circle$$
$$t \mapsto (\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2})$$

2. Nodal cubic curve,

3. For any irreducible quadric hypersurface $X \subset \mathbb{P}^{n+1}_k$, X is rational iff $X(k) \neq \phi$

4. smooth cubic surfaces over $k = \overline{k}$ are rational.

The lectures will discuss the work of C.Voinsin,B.Hasset, : Chow groups of 0-cycles can be used to show many varieties are not rational.
Lemma 0.6.1. *$A_0(X)$ is a birational invariant for $X \in sm.proj.Var(k)$.*

Proof. For $char(k) = 0$, □

Lemma 0.6.2. *(moving lemma for 0- cycle)*
$X \in sm.proj.Var(k)$ and $\phi \neq U \subset X$, then $\forall$ 0-cycle α on X is rational equivalent to some one on U.

Proof. It sufficience to show when $\alpha \in X$ is closed points. $\exists$ curve $C \subset X$ st: $C \cap U \neq \phi$ and $\alpha \in C$. so for $z \in C - U$, the line bundle $\mathcal{O}_C(\alpha + mz)$ is of higher degree for $m \gg 0$. so $\exists s \in H^0(\mathcal{O}_C(\alpha + mz))$ □

Theorem 0.6.3. *(Decomposition of diagonal)*
$X \in sm.proj.Var(k)$. TFAE

1. *$\forall\ E/k$, $A_0(X) \longrightarrow A_0(X_E)$ is surjective.*
2. *$\forall\ E/k$, $deg : A_0(X_E) \longrightarrow \mathbb{Z}$ is isomorpphic.*
3. *$\exists$ decomposition in chow $A_0(X \times X)$:*

$$\triangle = X \times \alpha + B \tag{0.38}$$

where $\alpha \in A_0(X)$ is a 0-cycle and B supported on $S \times X$ for some proper closed subset $S \subset X$.

Proof. (1)⟹ (3): Take $E = k(X)$ and then $A_0(X) \to A_0(X_{k(X)})$ is surjective

$$\begin{array}{ccc} X_{k(X)} & \longrightarrow & X \\ \downarrow & & \downarrow \\ spec(k(X)) & \longrightarrow & spec(k) \end{array}$$

(3)⟹ (2): using correspondence $\triangle$ acting on $A_0(X)$, one has

$$\triangle_*\beta = \beta = (X \times \alpha + B)_*\beta = deg(\beta)\alpha + 0 \tag{0.39}$$

for any 0-cycle $\beta \in A_0(X)$. so $deg(\beta)\alpha = \beta$ in $A_0(X)$ where $B_*\beta = 0$ is due to moving lemma of 0-cycle.

□

0.6.2 Prof.Lawrence Ein's Distinguished lectures: Measures of irrationality of an algebraic variety

Definition 0.6.4. *$X \in Var(\mathbb{C})$ with $\dim X = n$,*

$$\begin{aligned} Irr(X) &:= \min\{d : X \dashrightarrow \mathbb{P}^n\} \\ Cov.gon(X) &:= \min\{d :\} \\ Conn.gon(X) &:= \min\{d :\} \end{aligned} \tag{0.40}$$

Remark:

- $Irr(X) = 1$ iff X rational
- $Irr(X) = \min\{d : [\mathbb{C}(X) : \mathbb{C}(t_1, \dots, t_n)] = d\ \}$
- $Cov.gon(X) = 1$

Theorem 0.6.5. *For a general smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $= d \geq$,*

then

$$Irr(X) = d - 1$$

0.7 My reading talk on localization techniques in counting theory

During the student's learning seminar, I plan to report E.Geber and Rahul.Pandharipande paper "localization of virtual class" and learn the technique of localization methods in curve counting.

0.7.1 History and Goal

- Bott residue formula:
  Let X be a n-dimensional smooth projective algebraic manifold and $E \to X$ holomorphic vector field of rank r. $v \in H^0(X, TX)$ is a holomorphic vector field on X. $P : \mathbb{C}^{r\times r} \to \mathbb{C}$ is invariant polynomial function, ie, $P(gAg^{-1}) = P(A),\ \forall g \in GL(r)$, eg, $det,\ Trace$.

  $$\int_X P(\frac{\sqrt{-1}}{2\pi}\Theta) = \sum_{v(x)=0} \frac{P(A_x)}{\det A_x}) \tag{0.41}$$

  where for a zero x of v, choose a local coordinate $(U, z_1, ..., z_n)$, one can write v in the coordinate as

  $$v(z) = \sum_{1\le i,j\le n} a_{ij} z_i \frac{\partial}{\partial z_j}$$

  and define

  $$A_x := (a_{ij})_{n\times n}$$

  It's easy to check that different choices of coordinate result in the conjugation of A_x, so RHS is well-defined.

  Proof. idea of proof: □

  eg: $X := \mathbb{P}^n$ and $v =$

- Atiyah-Bott-Duistermaat-Hecknan

  Let G be a connected Lie group acting X a compact smooth manifold with fixed locus $X^G = \bigsqcup_i X_i$, $X_i \subset X^G$ connected component. Then

  $$\int_X \alpha = \sum_i \int_{X_i} \frac{\iota_i^* \alpha}{e(N_i)} \tag{0.42}$$

  where $\iota_i : X_i \hookrightarrow X$ is natural inclusion and N_i is normal bundle. It also has equivariant version

- Localization in counting.
  Let X be a smooth DM stack with $\mathbb{C}^* \curvearrowright X$ and $X^{\mathbb{C}^*} = \bigsqcup_i X_i$, $\iota : X^{\mathbb{C}^*} \hookrightarrow X$ then

$$[X]^{vir} = \iota_*(\sum_i \frac{[X_i]^{vir}}{e(N_i^{vir})}) \tag{0.43}$$

  where N_i is the virtual normal bundle.

  Proof. idea of proof:

  □

- Topological vertex:

0.7.2 Main idea of proof

0.7.3 Computation Examples

.

1. **Computation of multiple cover contributions.**

universal stable map

$$\begin{array}{ccc} C & \xrightarrow{\mu} & \mathbb{P}^r \\ \pi \downarrow & \nearrow & \\ \overline{M}_{g,n}(\mathbb{P}^r, d) & & \end{array}$$

Theorem 0.7.1. *(Farber-Pandharipande)*

$$\begin{aligned} GW_{g,d} < l_1, ..., l_n >= & \int_{[\overline{M}_{g,n}(\mathbb{P}^1,d)]^{vir}} \prod_{j=1}^{n} ev_j^* H^{l_j} \\ = & \sum_{\Gamma} \frac{1}{|A_\Gamma|} \int_{\overline{M}_\Gamma} \frac{\prod_{[n]} \lambda}{e(N_\Gamma^{vir})} \end{aligned} \tag{0.44}$$

Proof. □

Theorem 0.7.2. *(Morrison-Aspitoll, Farber-Pandharipande)*

$$\int_{\overline{M}_g(\mathbb{P}^1,d)} c_{top}(R\pi_* \mu^* N) = \begin{cases} \frac{1}{d^3} & if g = 0, \\ \frac{d}{12} & if g = 1, \\ \frac{|\chi(M_g)|}{} = & if g > 1, \end{cases} \tag{0.45}$$

Proof. □

2. **Toric CY 3-fold.**

$T \curvearrowright X$, the moment map

$$\mu_T : X \to \mathfrak{t}^*$$

has image $\mu_T(X)$ as a polytope.

0.7.4 Pandeharipande-Pixton's work on GW/DT correspondence for CI CY 3-fold

This section is based on my reading notes on Pandeharipande-Pixton's beautiful paper **??** and the two talks given during the AG program.

1. General conjecture of MNOP on the correspondence:

The (free energy amptitude at g) reduced GW potential for X

$$F_{GW}(u,v) := \sum_{g\geq 0}\sum_{0\neq\beta} N_{g,\beta}u^{2g-2}v^{\beta} \tag{0.46}$$

$$Z_{GW}(u,v) := exp(F_{GW}(u,v)) = 1 + \sum_{0\neq\beta} Z_{GW}(u)_{\beta}v^{\beta} \tag{0.47}$$

where

$$N_{g,\beta} := deg([\overline{M}_g(X,\beta)]^{vir}) = \int_{[\overline{M}_g(X,\beta)]^{vir}} 1 \in \mathbb{Q}$$

The reduced DT potential is

$$Z_{DT}(q,v) := \frac{\sum\limits_{\beta,n} T_{n,\beta}q^n v^{\beta}}{\sum\limits_{\beta,n} T_{n,0}q^n} \tag{0.48}$$

where $T_{n,\beta} := \int_{[I_n(X,\beta)]^{vir}} 1 \in \mathbb{Z}$.

By the work of MNOP, Jun Li, K.Behrend, it is known degree 0 DT is

$$M(-q)^{\chi(X)} = (\frac{1}{\sum\limits_{n\geq 1}(1-(-q)^n)})^{\chi(X)}$$

where $M(q) := \frac{1}{\sum\limits_{n\geq 1}(1-q^n)}$ is so-called Mcmahon function.

The conjectural relation for GW/DT correspondence is after change of variable $q \mapsto -e^{\sqrt{-1}u}$

$$Z_{DT}(-e^{\sqrt{-1}u}, v) = Z_{GW}(u,v) \tag{0.49}$$

2. what are known before Pandeharipande-Pixton ?

- GW/DT correspondence holds for Toric CY 3-fold by MNOP via localization techniques, topological vertex.
- DT theory for local curve (ie, total space of rank 2 bundles over a curve) by Br - via .
-

3. Main result of Pandeharipande-Pixton and main idea of the proof

Theorem 0.7.3. *(Pandeharipande-Pixton 2017, see* **??***)*
If $X \subset \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_r}$ *is a Fano or CY 3-fold of complete intersection, then*

$$\begin{aligned} &(-q)^{\frac{d_\beta}{2}} Z_{PT}(q, \tau_{\alpha_1}(\gamma_1) \cdot \ldots \cdot \tau_{\alpha_l}(\gamma_l))_\beta \\ = &(\sqrt{-1}u)^{d_\beta} Z_{GW}(u, \tau_{\alpha_1}(\gamma_1) \cdot \ldots \cdot \tau_{\alpha_l}(\gamma_l))_\beta \end{aligned} \tag{0.50}$$

where $d_\beta := \int_\beta c_1$ *under change of variable* $q \mapsto -e^{\sqrt{-1}u}$.

Proof. **Sketch:** The main idea is to use the typical computational techniques: Deformation and Degeneration.

□

0.7.5 what can we do next ?

Some problem

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