1 Introduction

K3 surface is introduced by A. Weil 1958, named after three mathemacians Kummer, Kahler, Kodaira and a mountain K2.

The main goal of the minicourse is to give a sketch of construction of the moduli space

$$\mathcal{F}_{2l} := \{(X, L) \text{ quasi-polarised K3 of degree } 2l\}/\sim \;\cong\; \Gamma \setminus G/K$$

where $G = SO(2, 19)$ is an orthogonal group , $K = SO(2) \times SO(19)$ is a maximal subgroup and $\Gamma \leq G$ is an arithmetic subgroup. Here the homogenous space G/K can be realised as a hermitian symetric domain of Type IV.

The plan for the mini-course will be

1. Lect 1: Overview and introduction for K3 in various subjects, stating the main result for moduli spaces of K3 surfaces and provide some examples.
2. Lect 2: More examples and Basic topological and geometrical properties of K3 surfaces.
3. Lect 3: Hodge theory for K3 and Torelli theorem
4. Lect 4: Cone structure for K3.
5. Lect 5: Moduli problem for K3. The existence of coarse moduli space.
6. Lect 6: The moduli space is a locally symmetric space via period map. Maybe some cycle theory problems will be mentiond.

A nice reference for the minicourse is Daniel Huybrechts' book

Lectures on K3 surfaces. Cambridge University Press. 2016

2 Geometry of K3 surfaces

Definition 2.1. *A complex K3 surface is a* 2*-dimensional simply connected compact* $\mathbb{C}$ *surface with* $\omega_X \cong \mathcal{O}_X$*.*

Or symplectic geometric definition: a 2*-dimensional simply connected compact* $\mathbb{C}$ *surface with a nowhere vanishing holomorphic* 2*-form* $\sigma \in H^0(X, \Omega_X^2)$ *such that* $H^0(X, \Omega_X^2) = \mathbb{C}\sigma$*.*

More algebraic-geometric definition: A K3 X *over* k *is a smooth projective surface* X *over* k *such that*

$$\omega_X \cong \mathcal{O}_X, \quad H^1(X, \mathcal{O}_X) = 0.$$

Remark 2.2. *A deep result of Siu (see [1]) says that a complex K3 surface must be a Kahler surface and so one may view it as a part of the definition, thus the results in Kahler geometry can be applied to K3 directly, for example, the Hodge decomposition etc.*

Remark 2.3. *Symplectic geometric definition can be generalised to higher dimension and the manifold is known as irreducible holomorphic symplectic manifold. Using Yau's famous solution of Calabi conjecture, this is equivalnent to the definition of hyperkahler manifold* (M, g, I, J, K)*.*

Remark 2.4. *Symplectic geometric and complex geometric definition are equivalent. A nowhere vanishing holomorphic 2-form σ will induce isomorphism of holomorphic vector bundle*

$$\sigma: \ T_X \to \Omega_X, \quad \text{via } v \ \mapsto \ \sigma(v, -).$$

As X is simply connected, $\wedge^2 T_X \cong \wedge^2 \Omega_X$ will imply $\omega_X \cong \mathcal{O}_X$.

On the other hand, $\omega_X = \wedge^2 \Omega_X \cong \mathcal{O}_X$ will imply

$$H^0(X, \Omega_X^2) = H^0(X, \omega_X) = H^0(X, \mathcal{O}_X) \cong \mathbb{C}.$$

For the algebraic definition, if $k = \mathbb{C}$, then the analytic space X^{an} (under Serre's GAGA) of an algebraic K3 X/k is complex K3 surface. While a complex K3 may be not an algebraic K3. See Remark (2.6).

In this minicourse, we mainly deal with the complex algebraic K3 surfaces. So a K3 always means a complex algebraic K3 surface unless otherwise specified.

Before exploring the examples, let us fix some convention. For smooth projective variety X over $\mathbb{C}$, each Weil divisor D on X will produce an invertible sheaf $\mathcal{O}_X(D)$ of rank 1, i.e, a line bundle. We may do not distinguish them in the mincourse.

The construction of explicit examples of K3 surfaces involves many beautiful geometry, we just list the following

Example 2.5 (Kummer K3s)**.** *Let $A = \mathbb{C}^2/\Lambda$ be an Abelian variety and*

$$\iota : A \to A, \quad z + \Lambda \ \mapsto \ -z + \Lambda$$

be an involution. It is easy to see its fixed locus is given by

$$A^\iota := \{(z_1, z_2) \mid (2z_1, 2z_2) \in \Lambda\}$$

and the quotient surface A/ι is surface with 16 isolated singular points of type A_1 (i.e., locally isomorphic to the germ $x^2 + y^2 + z^2 = 0$). Let $X \to A/\iota$ be the minimal resolution. Alternatively, one can take blowup $\pi : \widetilde{A} = Bl_{A^\iota} \to A$ the fixed locus A^ι and the involution ι will be lifted to $\widetilde{\iota} : \widetilde{A} \to \widetilde{A}$ such that $\pi \circ \widetilde{\iota} = \iota \circ \pi$. Then $X \cong \widetilde{A}/\widetilde{\iota}$ with the following diagram

$$\begin{array}{ccc}
\widetilde{A} = Bl_{A^\iota} & \xrightarrow{\ \pi\ } & A \\
\downarrow{\scriptstyle \widetilde{\iota}} & & \downarrow{\scriptstyle \iota} \\
\widetilde{A} & \xrightarrow{\ \pi\ } & A \\
\downarrow{\scriptstyle q} & & \downarrow \\
X & \longrightarrow & A/\iota
\end{array}$$

Denote the $E_1, \cdots, E_{16}$ the exceptional divisor of π and $C_i := q(E_i) \subset X$. The quotient map $q : \widetilde{A} \to X$ is a double cover brached along the curve $C_1 + \cdots + C_{16}$ and $q^*(C_i) = 2E_i$ for $1 \leq i \leq 16$.

Claim: X is s smooth surface. Note that q is étale outside $C_1 \sqcup \cdots \sqcup C_{16}$. Thus, it is sufficent to show X is smooth at $x \in C_i$ for any $1 \leq i \leq 16$. Indeed, assume $q(y) = x$, then locally around $y \in \widetilde{A}$ under suitable coordinate (u, v), the involution $\widetilde{\iota}$ is of the following form

$$(u, v) \ \mapsto (-u, v)$$

where $v = 0$ the the local equation for the rational curve E_i. Thus, the germ for x is given by $\mathrm{Spec}((k[u, v])^{\widetilde{\iota}}) = \mathrm{Spec}(k[u^2, v]) \cong \mathbb{A}_k^2$, which is smooth.

By the blowup, we get

$$K_{\widetilde{A}} = \pi^* K_A + \sum E_i = \sum E_i$$

By the double covering structure, we get

$$q_* \mathcal{O}_{\widetilde{A}} = \mathcal{O}_X \oplus \mathcal{O}_X(-\sum C_i), \quad K_{\widetilde{A}} = q^* K_X + \sum E_i$$

These formulas imply $\omega_X \cong \mathcal{O}_X$. Meanwhile, as $\iota^*(dz_i) = -dz_i$ for $i = 1, 2$, then

$$H^1(X, \mathcal{O}_X) \hookrightarrow (H^1(\widetilde{A}, \mathcal{O}_{\widetilde{A}}))^{\widetilde{\iota}^*} = (H^1(A, \mathcal{O}_A))^{\iota^*} = 0$$

The above discussion shows that X is a K3, which is called Kummer K3 . Moreover, from the construction, we have

$$17 \leq \rho(X) = \rho(A) + 16$$

and the Neron serveri group $NS(X)$ contains a sublattice $\langle -2 \rangle^{\oplus 16}$ spanned by the rational curves C_i.

This construction still works in higher dimension Abelian variety A/k with $\mathrm{char}(k) \neq 2$ and the variety X obtained in this way is called Kummer variety.

Remark 2.6. *In the above example, if we just take $A = \mathbb{C}^2/\Lambda$ to be a complex tori (eg, for generic lattice $\Lambda \subset \mathbb{C}^2$ so that there is no positive definite Hermitian form H on $\mathbb{C}^2$ whose imginary part E takes $\mathbb{Z}$-value on Λ, by Kodaira's embedding theorem, such complex surface is not projective), then the surface X we get is just complex K3, not necessary to be an algebraic K3.*

Remark 2.7. *Kummer K3 is an very important example in K3 surface. For example, in Šapiro- Šafarevič's proof of Torelli theorem of K3, they use the density of period points of kummer K3 to reduce the problem on how to recover a Kummer surface from its period point. Another important application of this geometric contrction is Mori-Mukai's theorem on the existence of rational curves in any ample class H on a K3 surface X. Indeed, they first produce a rational curve on an explicit Kummer K3 X associated Abelian surface $A = E_1 \times E_2$ where*

$$E_1 = \mathbb{C}/(2d+5)\mathbb{Z} + \sqrt{-1}\mathbb{Z},\ E_1 = \mathbb{C}/(\mathbb{Z} + \sqrt{-1}\mathbb{Z}), \quad \text{with isogeny } E_1 \xrightarrow{\phi} E_2$$

whose graph $\Gamma_\phi \subset A$ will give a rational curve in X. Then they use deformation theory of K3 with a curve to show the general result.

Example 2.8. *(K3s as an anti-canonical section of a Fano 3-fold) Let Y be a smooth projective 3-fold with $\mathrm{Pic}(Y) = \mathbb{Z}[-K_Y]$ and $X \in |-K_Y|$ is smooth, then by adjunction formula,*

$$\omega_X = (\omega_Y \otimes \mathcal{O}_Y(X))|_X = \mathcal{O}_X.$$

By taking the cohomology of

$$0 \to \mathcal{O}_Y(-X) \to \mathcal{O}_Y \to i_*\mathcal{O}_X \to 0$$

one can see that $H^1(X, \mathcal{O}_X) = 0$ and thus X is a K3 surfaces. From the classification theory of Fano 3-folds of Mukai-Mori, there are only finitely many such Fano 3-folds whose explicit construction given Mukai. See the table 1.

ℓ	General members in $F_{2\ell}$	Fano 3-fold Y
2	surface in $\lvert\mathcal{O}_Y(4)\rvert$	$\mathbb{P}^3$
3	surface in $\lvert\mathcal{O}_Q(3)\rvert$	quadratic 3-fold $Q \subset \mathbb{P}^4$
4	surface in $\lvert\mathcal{O}_Y(2)\rvert$	$Y = Q_1 \cap Q_2 \subset \mathbb{P}^5$

Tabel 1: Mukai Model for (X, L)

One of the application of this geometric construction is that one can show the unirationality of moduli spaces of quasi-polarised K3 surfaces in these cases.

Example 2.9. *(K3s from covering construction) Let Y be a del pezzo surface or $\mathbb{F}_4$ and $C \in |-2K_Y|$ be a smooth curve. Consider the double cover*

$$\phi: \ X \to Y$$

branched along the curve C. Huiwitz formula implies $\omega_X = \phi^*(\omega_Y \otimes \mathcal{O}_Y(\frac{1}{2}C)) = \mathcal{O}_X$ and as branched locus has real dimension ≥ 2, then $\pi_1(X) = \pi_1(Y) = 1$. Thus X is a K3. This covering construction will produce an involution $\tau : X \to X$ by exchanging the two sheets. Such involution is called anisymplectic involution, i.e, $\tau^*\sigma = -\sigma$. The K3 surfaces with anisymplectic involution can be classified according to their Neron-Serveri group $NS(X)$.

Remark 2.10. *The above two examples of K3 are related to Fano geometry. Especially, the moduli spaces of such K3s are related to moduli spaces of K-stable Fano objects.*

References

[1] Y. T. Siu. Every $K3$ surface is Kähler. *Invent. Math.*, 73(1):139–150, 1983.