0.1 Basic invariants of K3

We will introduce basic invariants for algebraic surfaces, which are very useful. Let's recall some very useful tools in complex geometry and algebraic geometry.

Theorem 0.1 (Hirebruch-Riemann-Roch formula). *Let X be a smooth projective surface and E is a complex vector bundle (or coherent sheaf) on X, then*

$$\chi(X,E) = h^0(E) - h^1(E) + h^2(E) = \int_X ch(E)td(T_X) \tag{1}$$

where $h^i(E) := \dim_{\mathbb{C}} H^i(X,E)$ is dimension of i-th Doeubleat (or sheaf cohomology) cohomology space In particular, for line bundle L (or a divisor D) on a K3 surface X, we have

$$\chi(X,L) = \frac{1}{2}L^2 + \int_X td_2 = \frac{1}{2}L^2 + \frac{c_1^2 + c_2}{12} = \frac{1}{2}L^2 + 2$$

Theorem 0.2 (Serre duality). *Let E be a vector bundle on a smooth projective surface X, then there is isomorphism of $\mathbb{C}$-vector spaces*

$$H^i(X,E) \cong H^{2-i}(X, E^* \otimes \omega_X)$$

for any $0 \le i \le 2$.

Theorem 0.3 (Kawamata-Viewheg Vanishing theorem). *Let L be a big [1] and nef line bundle on a smooth projective surface X, then*

$$H^i(X, \omega_X \otimes L) = 0, \quad i > 0.$$

Recall the Picard group $\mathrm{Pic}(X)$ of algebraic variety is isomorphism classes of line bundles on it and the Neron-Severi group $NS(X)$ is defined as

$$\mathrm{NS}(X) := \mathrm{im}(\mathrm{Pic}(X) \xrightarrow{c_1} H^2(X,\mathbb{Z})).$$

The rank $\rho(X) := rank(\mathrm{Pic}(X))$ is called the Picard number of X. By taking the cohomology of the exponential sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0, \tag{2}$$

we have $NS(K3) \cong \mathrm{Pic}(K3)$. Note that $\mathrm{Pic}(K3)$ is torsion free. Indeed, the sequence (2) also implies $\mathrm{Pic}(K3) \hookrightarrow H^2(K3,\mathbb{Z})$ and thus it is sufficient to show $H^2(X,\mathbb{Z})$ is torsion free for a K3 X. One way to see this is to use the sequence (2) to get

$$0 \to H^1(X, \mathcal{O}_X^*) = \mathrm{Pic}(X) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \xrightarrow{r} H^2(X, \mathcal{O}_X) = \mathbb{C}$$

Thus, if $H^2(X,\mathbb{Z})$ has a n-torsion class $\alpha \neq 0$, then $n \cdot r(\alpha) = r(n \cdot \alpha) = r(0) = 0$ and thus $r(\alpha) = 0$ as $\mathbb{C}$ has no torsion. This shows $\alpha \in \mathrm{im}(c_1)$ which can be lifted as a torsion line bundle L with $L^2 = 0$, then Riemann-Roch formula shows $h^0(L) + h^0(L^\vee) \ge 2$. We may assume $h^0(L) \ge 1$, i.e., L is effective. If the section $s \neq 0 \in H^0(X,L)$ has zeros, then so is any power $s^m \in H^0(X, L^m)$ and thus all L^m can not be trival, contradicting that L is torsion.[2]

Another view point is via covering trick: if $H^2(X,\mathbb{Z})$ has a n-torsion element, then there is étale covering $Y \to X$ of degree $n > 1$. As $\omega_Y \cong \mathcal{O}_Y$, then the Noether formula and serre duality $h^2(Y,\mathcal{O}_Y) = h^0(Y,\mathcal{O}_Y)$ implies

$$h^0(Y,\mathcal{O}_Y) - h^1(Y,\mathcal{O}_Y) + h^2(Y,\mathcal{O}_Y) = 2 - h^1(Y,\mathcal{O}_Y) = \chi(Y,\mathcal{O}_Y) = n\chi(X,\omega_X) = 2n,$$

which is a contradiction.

Theorem 0.4 (Hodge index theorem for algebraic surfaces). *Let X be a smooth projective surface and D a divisor on X such that $D^2 > 0$. If E is a divisor on X with $E.D = 0$, then $E^2 \le 0$ and $E^2 = 0$ if and only if $E \equiv 0$.*

Now we apply the above tools to show

Proposition 0.5 (Basic invariants of a K3). *Let X be a complex K3, then*

1. $H^1(X,\mathbb{Z}) = H^3(X,\mathbb{Z}) = 0$.
2. *the* $(\mathrm{NS}(X), \cup)$ *is a lattice of signature* $(1, \rho - 1)$ *with* $\rho \le 20$.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 1 & & 20 & & 1 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 0 & & 1 & & 0 & & 0
\end{array}.
$$

3. Hodge diamond of X is

Proof. The first assertion can be proved by showing $H^1(X,\mathbb{Z})$ torsion free, which can be done via covering trick.

The signature of $(\mathrm{NS}(X),\cup)$ is a direct consequence of Hodge index theorem (0.4).

To compute the Hodge diamond, first, from Riemann-Roch formula (1) for $\mathcal{O}_X$ and Serre duality, we have

$$h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X) = 2 = \frac{K_X^2 + c_2}{12},$$

So $c_2(X) = e_{top}(X) = 24 = 2b_0 - 2b_1 + b_2$ by poncaré duality. As connecteness and simply-connecteness imply $b_0 = 1, b_1 = 0$, we have $b_2 = 22$ at once. Then by the Hodge decomposition

$$H^2(X,\mathbb{C}) = H^0(X,\Omega_X^2) \oplus H^1(X,\Omega_X^1) \oplus H^2(X,\mathcal{O}_X),$$

and Hodge symmetry $H^2(X,\mathcal{O}_X) \cong H^0(X,\Omega_X^2) \cong \mathbb{C}$, we get $22 = b_2 = 2h^{2,0} + h^{1,1} = 2 + h^{1,1}$. □

Remark 0.6. *For algebraic K3 surface X over filed k of $\mathrm{char}(k) = p > 0$, it may happen that $\rho(X) = rank\,\mathrm{Pic}(X) = 22$, which is called supersingular K3. The main reason is that over $\mathbb{C}$, there is natural Hodge decomposition for $H^2(X,\mathbb{C})$ while in $\mathrm{char}(k) = p > 0$ case, there is no such decomposition for $H^2_{\acute{e}t}(X_{\overline{k}}, \overline{\mathbb{Q}}_\ell)$.*

Remark 0.7. *As $T_X \cong \Omega_X$, thus the Hodge diamond of a K3 shows*

$$h^0(X,T_X) = 0,\ h^1(X,T_X) = 20,\ h^2(X,T_X) = 0.$$

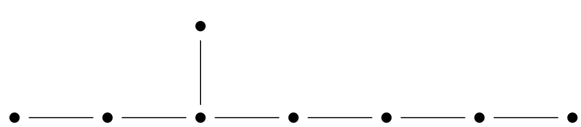
By the deformation theory, the 1st order deformation space of complex structure is $H^1(X,T_X)$ which is 20-dimensional complex space and the obstruction space to deform the complex structure is $H^2(X,T_X)$ which is trivial. This calculation shows the moduli space of complex structures of K3 surface is 20-dimensional. $h^0(X,T_X) = 0$ implies the automorphism group $\mathrm{Aut}(X)$ of K3 is 0-dimensional since each $\phi \in \mathrm{Aut}(X)$ can be viewed as a subvariety $\Gamma_\phi \subset X \times X$, whose normal bundle $N_{\Gamma_\phi/X\times X} \cong T_X$, As the deformation space of Γ_ϕ in $X \times X$ is just $H^0(\Gamma_\phi, N_{\Gamma_\phi/X\times X}) \cong H^0(X,T_X) = 0$, there is no moduli in $\mathrm{Aut}(X)$.

0.2 Hodge theory of K3 surfaces

Proposition 0.8. *$(H^2(X,\mathbb{Z}),\cup)$ is a lattice isomorphic to $U^3 \bigoplus E_8^2(-1)$ where U is hyperbolic lattice, i.e, a free $\mathbb{Z}$-module of rank 2 and the gram matrix under a basis e, f is of the form*

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and E_8 is the definite lattice associated to the diagram



[1] L is big means $\lim_{m\to\infty} \frac{h^0(L^m)}{m^2} > 0$

[2] This argument works for K3 over other filed.

Proof. First, the $H^2(X,\mathbb{Z})$ is a free $\mathbb{Z}$-module of rank 22. The cup product $\cup$ produce a bilinear form on $H^2(X,\mathbb{Z})$. As the second Stiefel-whitney class

$$w_2(X) \equiv c_1(X) = 0 \mod 2,$$

then $\alpha^2 \equiv 0 \mod 2$ for any $\alpha \in H^2(X,\mathbb{Z})$, i.e., $(H^2(X,\mathbb{Z}),\cup)$ is an even lattice Wu's result. Then by the Hirzebruch's signature formula on the middle cohomology of $2n$-dimensional real manifold, we have

$$\tau(X) = b_2^+ - b_2^- = \int_X \frac{1}{3}p_1(X) = \int_X \frac{c_1^2 - 2c_2}{3} = -16$$

where $p_1(X) = -c_1(T_X \oplus \overline{T_X})$ is the 1st Pontrayin class of X as a real 4-dimensional smooth manifold. The Poincaré duality implies $(H^2(X,\mathbb{Z}),\cup)$ is a unimodular lattice. Applying the classification theorem for indefinite even lattice, there is a unique lattice even indefinite lattice $U^3 \bigoplus E_8^2(-1)$ with signature $(3,19)$. This finishes proof that $(H^2(X,\mathbb{Z}),\cup) \cong U^3 \bigoplus E_8^2(-1)$. $\square$

We recall the general Hodge theory in algebraic geometry

Definition 0.9. *A (pure) $\mathbb{Z}$-Hodge structure of weight n is a free $\mathbb{Z}$-module V with decomposition*

$$V_{\mathbb{C}} = V \otimes_{\mathbb{Z}} \mathbb{C} = \underset{p+q=n}{\oplus} V^{p,q}, \text{ such that } \overline{V^{p,q}} = V^{q,p}.$$

Similarly, one can define $\mathbb{Q}$-Hodge structure.

Remark 0.10. *Equivalently due to Deligne's insight, one can define a $\mathbb{Q}$-Hodge structures on a $\mathbb{Q}$-vector space V of weight n by the representation of Deligne torus $h : \mathbb{S}(\mathbb{R}) \to GL(V_{\mathbb{R}})$ such that $h|_{\mathbb{R}^*} = t^n Id$ for any $t \in \mathbb{R}^*$. Indeed, as $\mathbb{R}$-algebraic group,*

$$\mathbb{S}(\mathbb{R}) \cong \mathbb{C}^* \cong \{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} | z = a + \sqrt{-1}b \in \mathbb{C}^*\}$$

and the complexication $h_{\mathbb{C}} : \mathbb{S}(\mathbb{C}) \to GL(V_{\mathbb{C}})$ for h will induce the action of $\mathbb{C}^* \times \mathbb{C}^*$ on $V_{\mathbb{C}}$ so that there is a decomposition

$$V_{\mathbb{C}} = \underset{\chi}{\oplus} V_{\chi} = \underset{(p,q)}{\oplus} V^{p,q}$$

where $\chi \in Hom(\mathbb{S}(\mathbb{C}), \mathbb{C}^*) \cong \mathbb{Z} \times \mathbb{Z}$ runs over the character of $\mathbb{S}(\mathbb{C})$ and $V_{\chi} := \{v \in V_{\mathbb{C}} \mid \lambda(v) = \chi(\lambda) \cdot v,\ \lambda \in \mathbb{S}(\mathbb{C})\}$ is the eigenspace. Under the embedding

$$\mathbb{R}^* \hookrightarrow \mathbb{S}(\mathbb{R}) \hookrightarrow \mathbb{S}(\mathbb{C}),\ t \mapsto \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix},\ z \mapsto (z, \overline{z})$$

$h|_{\mathbb{R}^*} = t^n Id$ for any $t \in \mathbb{R}^*$ implies $V^{p,q} = 0$ if $p+q \neq 0$, this gives weight n Hodge structure on V.

This perspective is related to give the Shimura datum in the definition of Shimura variety.

Remark 0.11. *Another equivalent definition is to use a decreasing filtration known as Hodge filtration, that is, a decreasing filtration $F^\bullet$*

$$F^{n+1} = \{0\} \subset F^n \subset \cdots \subset F^1 \subset F^0 = V_{\mathbb{C}}$$

on $V_{\mathbb{C}}$ such that

$$F^p \oplus \overline{F^{n+1-p}} = V_{\mathbb{C}}, \quad 0 \leq p \leq n.$$

Write $V^{p,q} := F^p \cap \overline{F^q}$, then $V_{\mathbb{C}} = \underset{p+q=n}{\oplus} V^{p,q}$.

This view point is easy to define the so-called period map via Griffith's variational Hodge structure theory.

Example 0.12. *Let X be an algebraic curve of genus g, then $H^1(X,\mathbb{Z})$ has a natural Hodge structure of weight 1 given by Hodge decomposition*

$$H^1(X,\mathbb{C}) \cong H^0(X,\omega_X) \oplus H^1(X,\mathcal{O}_X) \cong \mathbb{C}^{2g}.$$

In fact, there is one to one correspondence

$$\{\text{weight } 1 \text{ Pure Hodge structure}\} \leftrightarrow \{\text{ complex torus}\}.$$

Clearly, the Hodge decomposition will produce a weight 2 Hodge structure on $H^2(X,\mathbb{Z})$. Let $T_X := (\mathrm{NS}(X))^\perp \subset H^2(X,\mathbb{Z})$ be the orthogonal complement of Nero-Serveri group, known as transcendental lattice of a K3. Then T_X has a Hodge structure of weight 2. Moreover,

Proposition 0.13. *$T_X \subset H^2(X,\mathbb{Z})$ is the minimal primtive sub Hodge structure of weight 2 such that $(T)^{2,0} = H^{2,0} \cong \mathbb{C}$.*

Proof. Assume there is another such sub-Hodge structure T' so that

$$T' \subset T_X \subset H^2.$$

and there is a integral class $\alpha \in T - T'$. As $(T')^{2,0} = T^{2,0} = \mathbb{C}$, thus both $T^{2,0} \cap T'$ and $T^{2,0} \cap T$ are isomorphic to $\mathbb{Z}$. If $\alpha \in T_X \cap (T_X)^{(2,0)} \cong \mathbb{Z}$, there is a $\beta \in T' \cap (T_X)^{(2,0)} \cong \mathbb{Z}$ such that $n\beta = \alpha$ for some $n > 1$, this gives a torsion element $\beta \in H^2/T_X$. It gives a contradiction since T_X is primitive (i.e., H^2/T_X is torsion free). Then we may assume $\alpha \in T_X \cap (T_X)^{(1,1)} \subset H^2 \cap H^{1,1}$. By the Lefschetz $(1,1)$-classes theorem, $\alpha \in N$, which contradicts that $\alpha \perp N$. $\square$

Theorem 0.14 (Global Torelli)**.** *Two complex K3 X and Y are isomorphic if and only there is an Hodge isometry* [3]

$$\phi : \ H^2(X,\mathbb{Z}) \ \to \ H^2(Y,\mathbb{Z}).$$

If ϕ preserves Kahler cone, then $\phi = f^*$ for some isomorphic $f : Y \to X$.

Proof. We will explain the proof later. $\square$

Remark 0.15. *The philosophy of Torelli theorem is the Hodge structure of certain cohomology group $H^*(X,\mathbb{Z})$ of a smooth projective varieties X. The first example of Torelli theorem in history is that of smooth projective curves over $\mathbb{C}$, whose Hodge structure is of weight 1. In the case of weight ≥ 2, there are counter-examples for Torelli theorem.*

Remark 0.16. *The Torelli type theorem still holds for K3's higher dimensional generation —Hyperkahler mainfold, but it will be modified by replacing isomorphism classes to birational classes. Note that two birational K3 surfaces are indeed isomorphic.*

References

[3]that is, ϕ is a morphism of Hodge structure and preserves the cup product.