1 Moduli spaces via period map

1.1 Period domain of weight 2 polarised Hodge structure (PHS) of K3 type

Definition 1.1. *A ($\mathbb{Q}$-) PHS of weight 2 consists of vector space $V/\mathbb{Q}$ with symmetric bilinear form*

$$q:\ V \times V \to \mathbb{Q}$$

such that

1. *(Hodge filtration) A decreasing filtration $F^\bullet$*

$$F^3 = \{0\} \subset F^2 \subset F^1 \subset F^0 = V_\mathbb{C}$$

on $V_\mathbb{C}$ such that

$$F^p \oplus \overline{F^{3-p}} = V_\mathbb{C}, \quad 0 \le p \le 3.$$

2. *(Polarization compatibility) The form $\langle , \rangle$ on $V_\mathbb{C}$ defined by*

$$\langle v, w \rangle := (-1)^p q_\mathbb{C}(v, \overline{w}), \quad v, w \in V^{p,2-p} := F^p \cap \overline{F^{2-p}}$$

is a Hermintian form on $V_\mathbb{C}$ and positive definite on component $V^{p,2-p}$.

Or equivalently, Hodge filtration can be replaced by a $\mathbb{R}$ linear group representation

$$h:\ \mathbb{S}(\mathbb{R}) \to GL(V_\mathbb{R})$$

such that $h(t)v = t^2 v$ for any $v \in V_\mathbb{R}$ and $t \in \mathbb{R}^* \subset \mathbb{S}(\mathbb{R})$.

We can construct the classifying space of weight 2 PHS as

$$\mathcal{D} := \{F \in Fl(V_\mathbb{C}) \mid q_\mathbb{C}(v, w) = 0,\ v \in V^{p,2-p}, w \in V^{p',2-p'} \text{ for } p \ne p',\ \ \langle v, \overline{v} \rangle > 0\} \tag{1}$$

It is a open (under the Euclidean topology) subset of closed subvariety $\hat{\mathcal{D}}$ of the flag variety $Fl(V_\mathbb{C})$. Indeed, these spaces are homogeneous spaces. Let

$$G_\mathbb{C} := \{g \in GL(V_\mathbb{C}) \mid q_\mathbb{C}(g-, g-) = q_\mathbb{C}(-,-)\},\ P_\mathbb{C} := \{g \in GL(V_\mathbb{C}) \mid g(F^\bullet) = F^\bullet\}$$

where $F^\bullet \in \hat{\mathcal{D}}$ is reference point. Then $G_\mathbb{C}$ acts on $\hat{\mathcal{D}}$ transitively and P is just the stabilizer of $F^\bullet$, thus,

$$\hat{\mathcal{D}} \cong G/P.$$

Let $K := P \cap G$, then $K \le G_\infty$ is a maximal compact subgroup and $\mathcal{D} \cong G/K$.

Now we are interested in the classifying space $\mathcal{D}$ of weight 2 PHS of K3 type, that is, $h^{2,0} = 1$. In this situation, the period domain $\mathcal{D}$ can be realised as a Hermitian symmetric domain, that is,

$$\mathcal{D} = \{z \in \mathbb{P}\Lambda_\mathbb{C} \mid z^2 = 0,\ z.\overline{z} > 0\}^+.$$

Theorem 1.2 (Baily-Borel)**.** *For any arithmetic subgroup $\Gamma \le G$, the $\Gamma \setminus \mathcal{D}$ is a quasi-projective variety and there is a minimal compactification*

$$(\Gamma \setminus \mathcal{D})^* \cong \mathrm{Proj}(R(\Gamma \setminus \mathcal{D}, \lambda))$$

whose boundaries $(\Gamma \setminus \mathcal{D})^* - \Gamma \setminus \mathcal{D}$ consist of curves and points.

1.2 Period maps for family of K3

Let $\pi : (\mathfrak{X}, \mathcal{L}) \to B$ be a family of polarised K3, then we have the following data

$$\mathbb{V} := R^2\pi_*\mathbb{C} \otimes \mathcal{O}_B, \nabla : \mathbb{V} \to \mathbb{V} \otimes \Omega_B,\ \mathbb{F}^p := R^2\pi_*\mathcal{F}^p.$$

Denote the relative deRham complex

$$\Omega^\bullet_{\mathfrak{X}/B} := [\mathcal{O}_{\mathfrak{X}} \to \Omega^1_{\mathfrak{X}/B} \to \Omega^2_{\mathfrak{X}/B}],$$

which is quasi-isomorphic to $\pi^{-1}\mathcal{O}_T$, thus there is isomorphism

$$\mathbb{R}^2\pi_*\Omega^\bullet_{\mathfrak{X}/B} \cong \mathbb{R}^2\pi_*(\pi^{-1}\mathcal{O}_B) \cong R^2\pi_*\mathbb{C} \otimes \mathcal{O}_B = \mathbb{V}$$

Using the filtration $\mathcal{F}^\bullet$ on $\Omega^\bullet_{\mathfrak{X}/B}$ defined by $\mathcal{F}^p := \tau^{\geq p}\Omega^\bullet_{\mathfrak{X}/B}$, we get a filtration of holomorphic subbundles

$$\mathbb{F}^p := \mathrm{im}(\mathbb{R}^2\pi_*\mathcal{F}^p \to \mathbb{V})$$

of $\mathbb{V}$. Following Katz-Oda, using short exact sequence

$$0 \to \Omega^\bullet_{\mathfrak{X}/B}[-1] \wedge \pi^*\Omega^1_B \to \Omega^\bullet_{\mathfrak{X}}/(\pi^*\Omega^2_B \wedge \Omega^\bullet_{\mathfrak{X}/B}) \to \Omega^\bullet_{\mathfrak{X}} \to 0$$

and taking $\mathbb{R}\pi_*$, there is morphism known as Gauss-Manin connection

$$\mathbb{V} = \mathbb{R}\pi^2_*\Omega^\bullet_{\mathfrak{X}/B} \xrightarrow{\nabla} \mathbb{R}\pi^3_*(\Omega^\bullet_{\mathfrak{X}/B}[-1] \otimes \pi^*\Omega^1_B) = \mathbb{V} \otimes \Omega^1_B$$

satisfying $\nabla^2 = 0$.

These data are geometric realizations of so-called variational Hodge structure. We are going to define period map more generally for abstract VHS. Let's recall the variational polarised Hodge structure of weight 2 over T consists of $(\mathbb{V}, \nabla : \mathbb{V} \to \mathbb{V} \otimes \Omega_B, \mathbb{F}^\bullet, \mathcal{Q})$

- $\mathbb{V}$ is a holomorphic vector bundle over T such that its sheaf of sections are $\mathcal{V} \otimes_{\mathbb{Z}} \mathcal{O}_B$ for a $\mathbb{Z}$-coefficient local system $\mathcal{V}$.
- $\nabla : \mathbb{V} \to \mathbb{V} \otimes \Omega_B$ is a flat connection, i.e., it satisfies Leibnitz rule and $\nabla^2 = 0$.
- There is a holomorphic subbundle filtration $\mathbb{F}^2 \subset \mathbb{F}^1 \subset \mathbb{F}^0 = \mathbb{V}$ satisfying Griffith transverlity

  $$\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega_B$$

  for each p where $\mathcal{F}^p$ is the sheaf of hololomorphic sections of $\mathbb{F}^p$.
- $\mathcal{Q} :\ \mathcal{V} \otimes \mathcal{V} \to \mathbb{Z}$ is flat so that $(\mathbb{V}_b, \mathbb{F}^\bullet_b, \mathcal{Q}_b)$ is a PHS of weight 2 for any $b \in B$.

We may assume T is connected and $0 \in T$. Given a VPHS of K3 type $(\mathbb{V}, \nabla, \mathbb{F}^\bullet, \mathcal{Q})$, there is a map

$$p : T \to \Gamma \setminus \mathcal{D}$$

where $\Gamma := \mathrm{im}(\rho : \pi_1(T, 0) \to GL(\mathcal{V}_0)))$ is the monodromy group associated to $(\mathbb{V}, \nabla, \mathbb{F}^\bullet, \mathcal{Q})$. Indeed, for any path $\alpha : [0,1] \to T$ with $\alpha(0) =$ and $\alpha(1) =$, the parall transport along α w.r.t ∇ will define a $\mathbb{C}$-isomorphism

$$\alpha^* : \mathbb{V}_t \to \mathbb{V}_0$$

Thus, one can get a map

$$t \ \mapsto\ \alpha^*((\mathbb{V}_t)^{2,0}) \tag{2}$$

For another path $\beta : [0,1] \to T$ which gives another $\mathbb{C}$-isomorphism $\beta^* : \mathbb{V}_t \to \mathbb{V}_0$, one has [1]

$$\rho(\beta \circ \alpha^{-1}) = \beta^* \circ (\alpha^*)^{-1} :\ \mathbb{V}_0 \to \mathbb{V}_0$$

That is to say, the map

$$p :\ T \to \Gamma \setminus \mathcal{D},\ \ t \ \mapsto\ \alpha^*((\mathbb{V}_t)^{2,0}) \mod \Gamma \tag{3}$$

is well-defined.

[1] α^{-1} means the the same path as α but with opposite direction, so $\beta \circ \alpha^{-1}$ is a loop around $0 \in T$.

Theorem 1.3. *The period map p is holomorphic.*

Recall we have calculated deformation theory for complex structure on a given K3 X. As $H^0(X, T_X) = 0 = H^2(X, T_X)$, then we have Kuranishi space $B = Def(X)$ [2] with pointed family

$$(\mathfrak{X}, X) \xrightarrow{\pi} (B, 0) \tag{4}$$

such that $T_0 B = H^1(X, T_X) \cong \mathbb{C}^{20}$.

Theorem 1.4 (Local Torelli)**.** *The period map of the above families $p : B \to \mathcal{D} \subset \mathbb{P}H^2(X, \mathbb{C})$ is a local isomorphism.*

Proof. As p is a holomorphic map of two smooth complex manifold, It is sufficient to show the tangent map of the holomorphic map is bijective. Note that the tangent map of this local period map is just the composition of Kodaira-Spencer map of the family (4) and contraction with the nowhere vanishing σ

$$T_0 B \xrightarrow{ks} H^1(X, T_X) \xrightarrow{\sigma} T_{[H^{2,0}(X)]}\mathcal{D} = Hom(H^{2,0}(X), H^1(X, \Omega_X)) \tag{5}$$

Recall ks is just obtained by taking the cohomology of the short exact sequence

$$0 \to T_X \to T\mathfrak{X}|_X \to T_0 B \to 0$$

As $T_0 B = H^1(X, T_X)$ and in this case ks is isomorphism. □

Now the global Torelli theorem for polarised K3 surfaces says the global period map

$$p : \;\; F_{2l} \to \Gamma_{2l} \setminus \mathcal{D}_{2l}$$

is a bijective. This can be deduced from the global Torelli theorem for complex K3, which we are going to give a proof.

1.3 Global Torelli theorem for complex K3

Now we turn to explain the proof of global Torelli theorem for complex K3, which the proof can be generalised to give a proof of Verbisky's global Torelli theorem for HK. To do so, we need to define the moduli space of marked complex K3 surface,

$$N := \{(X, \phi : H^2(X, \mathbb{Z}) \xrightarrow{\sim}) \Lambda)\} / \sim$$

where $\Lambda = U^{\oplus 3} \oplus E_8^2$. Then we have period map

$$p : N \to \mathcal{D}_\Lambda = \{z \in \mathbb{P}\Lambda_{\mathbb{C}} \mid z^2 = 0, \; z.\overline{z} > 0\} \tag{6}$$

Remark 1.5. *One may compare this construction with the construction of moduli space M_g of curves of genus g via Teichmuller space*

$$T_g := \{(X, \phi : X \xrightarrow{\text{diffem}} \Sigma_g)\} / \sim$$

surjectivity of period map via twistor line

The twistor line builds on the hyperkahler structure $(I, J, K : T_{\mathbb{R}}X \otimes \mathbb{C} \to T_{\mathbb{R}}X \otimes \mathbb{C})$ of K3 X. Let

$$M = X \times S^2 = X \times \mathbb{P}^1 \to S^2 = \mathbb{P}^1$$

be the natural projection with fiber given by the complex structure

$$I_t := a \cdot I + b \cdot J + c \cdot K, \quad t = (a, b, c) \in S^2 \subset \mathbb{R}^3 \text{ with } a^2 + b^2 + c^2 = 1.$$

The map $M \to \mathbb{P}^1$ is called twistor family associated to hyperkahler structure (I, J, K).

Definition 1.6. *Let $W \subset \Lambda_{\mathbb{R}}$ be a 3-dimensional positive vector space (i.e., $\langle , \rangle|_W > 0$).*

[2] B can be taken to be a small disk in $H^1(X, T_X) \cong \mathbb{C}^{20}$

1. *the curve* [3] $l_W := \mathbb{P}W_{\mathbb{C}} \cap \mathcal{D}_\Lambda \subset \mathbb{P}W_{\mathbb{C}} = \mathbb{P}^2$ *is twistor line associated to* W.

2. *the twistor line* l_W *is called generic twistor line (GTL) if* $w^\perp \cap \Lambda = \{0\}$ *for any* $w \in W$.

We say two points $z, z' \in \mathcal{D}_\Lambda$ *are equivalent if there is a finitely many generic twistor lines* $l_{W_1}, \cdots, l_{W_n}$ *and a sequence of points* $z_0, z_1, \cdots, z_n$ *such that*

$$z_0 = z,\ z_n = z',\ z_i \in l_{W_i} \cap l_{W_{i+1}}.$$

Proposition 1.7. *Any two points* $z, z' \in \mathcal{D}_\Lambda$ *are equivalent.*

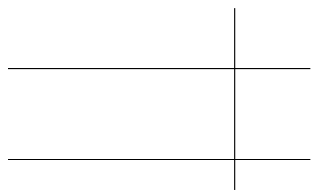
Proof. First, we claim that the set

$$\Sigma_z := \{z' \in \mathcal{D} |\ z' \sim z\}$$

is open for any $z \in \mathcal{D}_\Lambda$. Assume $P_z = \langle a, b\rangle$ [4] under isomorphism $\mathcal{D} \cong Gr^+(2, \Lambda_{\mathbb{R}})$. For any small pertubation z' of z under Euclidean topology, the set

$$\{c \in \Lambda_{\mathbb{R}} \mid \langle a, b, c\rangle, \langle a, b', c\rangle \text{ and } \langle a;, b;, c\rangle \text{ are positive 3-dimensional space in } \Lambda_{\mathbb{R}}\} \subset \Lambda_{\mathbb{R}}$$

is open where $P'_z = \langle a', b'\rangle$. Thus, we can choose a suitable $c \in \Lambda_{\mathbb{R}}$ such that there are three generic twistor lines $l_{\langle a,b,c\rangle}, l_{\langle a,b',c\rangle}$ and $l_{\langle a',b',c\rangle}$ so that



The claim will implies $\Sigma_z \subset \mathcal{D}$ is both open and closed. As $\mathcal{D}$ is connected, then $\Sigma_z = \mathcal{D}$. □

A very bad thing is that N is not Hausdorff. A teicnique introduced in [1] is the Hausdorff reduction of N. That is, $\overline{N} = N/\sim$ where $x \sim y$ if x and y are inseparable points. The very important property for the Hausdorff reduction is

Proposition 1.8. *The period map* $p : N \to \mathcal{D}$ *is uniquely factored through the natural quotient map* $N \to \overline{N}$

$$\begin{array}{ccc} N & \xrightarrow{p} & \mathcal{D} \\ \downarrow & \nearrow_{\overline{p}} & \\ \overline{N} & & \end{array}$$

such that $p([(X, \phi)]) = p([(X', \phi')])$ *if and only if* $[(X, \phi)], [(X', \phi')] \in N$ *are two inseparable points.*

Proof. see [1].

□

Remark 1.9. *Moreover,* $p([(X, \phi)]) = p([(X', \phi')])$ *will imply* X *is birational to* X'. *The argument is due to Huybrechts. As* $[(X, \phi)], [(X', \phi')] \in N$ *are two inseparable points, there are two sequence of marked complex K3s* (X_i, ϕ_i) *and* (X'_i, ϕ'_i) *with*

$$\lim_{i\to\infty} (X_i, \phi_i) = (X, \phi), \quad \lim_{i\to\infty} (X_i, \phi_i) = (X, \phi), \qquad X_i \xrightarrow{f_i \text{ isom}} X'_i.$$

By a result of Bishop, one get the limit cycle

$$\Gamma_\infty = Z \sum_i Y_i \subset X \times X'$$

of graph $\Gamma_{f_i} \subset X_i \times X'_i$ *of* f_i, *where* Z *is a correspondence from a birational map between* X *and* X' *and* Y_i *does not dominant* X *or* X'.

As any birational map of two smooth projective surface over $\mathbb{C}$ *with canonical divisors nef is indeed an isomorphism, so* $X \cong X'$. [5]

[3] This is a smooth plane conic.

[4] Here $\langle a, b\rangle$ means the $\mathbb{R}$-vector space spanned by the vectors $a, b \in \Lambda_{\mathbb{R}}$.

[5] Note that this fails for higher dimensions, say, there exists Mukai flop between HK of dimension ≥ 4.

Proposition 1.10 (Lifting of generic twistor line)**.** *For any (X, ϕ) and a generic twistor line $p([(X, \phi)]) \in l_W$, there is a unique morphism $l_W \to N$ such that*

$$\begin{array}{ccc} & & l_W \\ & \swarrow & \downarrow \\ \overline{N} & \xrightarrow{p} & \mathcal{D}_\Lambda \end{array}$$

where $\overline{N} = N/\sim$ is the Hausdorff reduction of N.

Proof. The idea is to lift GTL l_W as a real twistor family in moduli space.

By local Torelli theorem, we can take a local lifting, say a small disk $\in \Delta \subset l_W \cong \mathbb{P}^1$. That is , there is a family of marked complex K3

$$(\mathfrak{X} \to \Delta, (\phi_t : H^2(\mathfrak{X}_t, \mathbb{Z}) \to \Lambda)_{t\in\Delta})$$

The Hausdorff property of $\overline{N}$ implies such local lifting is unique.

As $\overline{p}(\Delta) \subset l_W$, then $\phi_t(\sigma_t) = \langle Re(\sigma_t), Im(\sigma_t)\rangle \subset W \subset \Lambda_\mathbb{R}$ defines a 2-dimensional positive space in W and take $\omega_t \in W$ so that

$$W = \langle Re(\sigma_t), Im(\sigma_t)\rangle \oplus \mathbb{R}\omega_t, \quad (\omega_t)^2 > 0.$$

Using generic properties of twistor lines l_W, we know that $\mathfrak{X}_t$ is a non-algebraic K3 and in this case, the Kahler cone of $\mathfrak{X}_t$ is just the cone

$$\{\alpha \in H^{1,1}(\mathfrak{X}_t, \mathbb{R}) \mid \alpha^2 > 0\}^+.$$

In other words, ω_t is a Kahler class on $\mathfrak{X}_t = (M, I_t)$. By Yau's solution of Calabi conjecture, there is Riemannian metric g_t together with another two complex structure J_t, K_t with

$$\omega_t(,) = g_t(I_t,), \quad \sigma_t = \omega_{J_t} + \sqrt{-1}\omega_{K_t}, \; J_tK_t = I_t = -K_tJ_t, .$$

Thus we get a twistor family $\mathcal{Y} \to \mathbb{P}^1$ associated to (I_t, J_t, K_t). Now we are going to show under the period map $\overline{p}$, $\mathbb{P}^1$ is identified with the realise GTL l_W.

□

Theorem 1.11. *The period map p is surjective.*

Proof. For any $z \in \mathcal{D}$, take a $(X, \phi) \in N$ and by Proposition , $p((X, \phi))$ and z can be connected by finitely many generic twistor lines connecting them. By Proposition , these lines are lifted to lines in $\overline{N}$, thus there is a pair (X', ϕ') with period $p((X', \phi')) = z$. □

Injectivity of period map

Proposition 1.12. *Let $N^0 \subset \overline{N}$ be a connected component, then $p : N^0 \to \mathcal{D}$ is a covering map*

Proof. By local Torelli theorem 1.4 and Hausdorff reduction 1.8, we only need to check the criterion 1.13. This can be done by checking

1. $B \subset p(C)$;
2. $\overline{B} - B \subset p(C)$.

These two results imply $\overline{B} \subset p(C)$ and so $\overline{B} = p(C)$ while $p(C) \subset \overline{B}$ is autmatical. For 1, by the proof of proposition 1.7, any two points x, y in B is connected by the connected by the component of $l_W \cap B$ the intersection of generic twistor line and B. As $B \cap p(C)$ nonempty, argue as in proposition 1.10 to get the local lifting in C.

□

Lemma 1.13 (A criterion of being covering space due to Verbitsky,Markman [1][Proposition A1)**.** *] Let $p : M \to D$ be a local homemorphism of two Hausdorff manifold. Then p is a covering space if and only if for any ball $B \subset D$ and each connected component C of $p^{-1}(\overline{B})$, $p(C) = \overline{B}$.* [6]

[6] That is, a continuous map $p : M \to D$ so that for any $y \in D$, there is a small open neighbothood $y \in U_y \subset D$ such that $p^{-1}(U_y) = \sqcup V_j$ and $p|_{V_j} : V_j \to U_y$ is homemorphism. This is a pure topological notion.

Proposition 1.14 (Characterization of Monodromy group)**.** *The monodromy group $Mon^2(X)$ for a K3 is isomorphic to $O^+(\Lambda)$.*

Proof. First, we recall some basic results about the orthogonal group $O(\Lambda)$. A key fact is that any $g \in$ is composition of finitely many reflective groups

$$s_\delta : \ v \ \mapsto v - 2\frac{v.\delta}{\delta^2}\delta, \quad \delta^2 \neq 0.$$

This defines a spinor norm $O(\Lambda) \to \mathbb{Z}/2\mathbb{Z}$ on $O(\Lambda)$. Set

$$1 \to O^+(\Lambda) \to O(\Lambda) \to \mathbb{Z}/2\mathbb{Z} \to 1.$$

Another key result is that each element of $O^+(\Lambda)$ is of the form $s_{\delta_1} \circ \cdots \circ s_{\delta_m}$ with $\delta_i^2 = -2$. Thus, it is reduced to show any reflection group s_δ comes from monodromy. monodromy group is deformation invariant. We may choose a complex K3 $X = (M, I)$ so that $\mathrm{Pic}(X) \cong \mathbb{Z}\delta$. As (-2)-curve on K3 is $\mathbb{P}^1$, we get a constraction morphism

$$X \to Y$$

which maps the (-2)-curve to a A_1 point on Y. As A_1 curve can be smoothed, and a small deformation of K3 is still K3, we get a deformation

$$\mathcal{Y} \to \Delta = \{t \mid |t| < 1\}$$

with $\mathcal{Y}_t$ a smooth complex K3 for $t \neq 0$ and $\mathcal{Y}_0 \cong Y$. Under base change $t \mapsto t^2$, we can get a new family $\mathfrak{X} \to \Delta$ with $\mathfrak{X}_0 \cong X$ and $\mathfrak{X}_t \cong \mathcal{Y}_t$ for $t \neq 0$. Then apply Picard-Lefschtz formula for the punctured family $\mathfrak{X}^* \to \Delta^*$, we get the image of monodromy representation

$$\pi(\Delta^*, t) \to O(H^2(\mathfrak{X}_t, \mathbb{Z})) \cong O(\Lambda)$$

is just given by s_δ. This shows $O^+(\lambda) \leq Mon^2(X)$.

Last, observe $-Id \notin Mon^2(X)$ and thus $O^+(\lambda) = Mon^2(X)$.

□

Proposition 1.15. *$p : N^0 \to \mathcal{D}$ is injectiive.*

Proof. This is purely a topological arguments. Note that the period domain $D = G/K$ is simply connected, then the cover space $p : N^0 \to \mathcal{D}$ from a connected manifold N^0 to D must be homemorphism. □

References

[1] Misha Verbitsky. Mapping class group and a global Torelli theorem for hyperkähler manifolds. *Duke Math. J.*, 162(15):2929–2986, 2013. Appendix A by Eyal Markman.