Notes on GIT and related topics

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2020/11/12

The reading note contains basics of GIT and some computational examples. A brief GIT construction of moduli of stable curves and stable maps will be given. At last, we discuss relations with other stability conditions in algebraic geometry and moment maps in symplectic geometry.

Contents

0.1 GIT Pholosophy

Lemma 0.1.1. *(Key lemma)If C is a Degline-Mumford stable curve of $g \geq 2$ over $\mathbb{C}$.then*

(i) $H^1(C, \omega_C^{\otimes n}) = 0$ *for* $n \geq 2$

(ii) $\omega_C^{\otimes n}$ *is very ample for* $n \geq 3$.

Proof. (i) comes from Vanishing Theorem. (ii) can be applied by Riemann-Roch and criteria

$$dim|\omega_C^{\otimes n}| - dim|\omega_C^{\otimes n} - P - Q| = 2$$

for $\forall$ $P, Q \in C$ □

Remark 0.1.2. *By the the lemma, choose $n = 3$, universal for all stable curves, then we have the embedding*

$$C \hookrightarrow \mathbb{P}^{5g-6}$$

then we can consider the Hilbert scheme $H_{p,5g-6}$ which parameterizes all i-dimensional closed subscheme in $\mathbb{P}^{5g-6}$.Then take its GIT quotient.This is a sample of GIT approach to construction of moduli space. The advantage of this approach is that projectivity of moduli spaces is obvious.

0.2 Basic setting of GIT under Reductive group action

Definition 0.2.1. *We call a group scheme G an* ***algebraic linear group*** *over k if it is a smooth closed subgroup scheme of $GL(n)$ over k. or equivalently, it is an affine smooth group scheme of finite type over k.*

Reductive group:
An algebraic group is **semi simple** if radical

$$R(G) := \{g \in G : \ (g-1)^r = 0 \, , r \in \mathbb{N} \} = \{e\}$$

ie, its maximal closed connected normal solvable subgroup is trival.
is **reductive** if its unipotent radical $R_u(G) = \{e\}$, ie, unipotent element in $R(G)$. in particular, over $k = \mathbb{C}$, a G is reductive iff $G = K \otimes_{\mathbb{R}} \mathbb{C}$ is a complexition of

Lemma 0.2.2. *If G is reductive and R is finite generated k-algebra, then R^G is also a finite generated k-algebra.*

Proof. □

0.2.1 Affine GIT

Let $X = \texttt{Spec}(R)$ be an affine scheme and G affine algebraic group.

$$X /\!\!/ G := \texttt{Spec}(R^G) \tag{0.2.1}$$

0.2.2 Projective GIT

The (moduli) meaning of GIT quotient is that it is a geometric quotient.

Definition 0.2.3. *Let $p : X \to Z$ be a G-morphism over k. we call p is a*

1. *categorical quotient if $\forall$ G-morphism $f : X \to Y$ factor though p, ie,*

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "\forall\ f"'] & Z \arrow[dl, "\exists"] \\ Y & \end{tikzcd}$$

2. *geometric quotient if and*
   - *For $\forall z \in Z$, $\exists$ affine $z \in U \subset Z$, st:*
   - *$\forall$ two disjoint open $V_1, V_2 \subset X$, the images $p(V_1), p(V_2) \subset Z$ are still disjoint open.*

Theorem 0.2.4. 1. *The standard GIT quotient $X \to X/\!\!/G$ is a geometric quotient.*

2. *$p : X^{ss} \to X/\!\!/G$, $p^{-1}([x])$ contains a unique closed orbit of a semi-stable point. $X/\!\!/G$ parameterizes S-equivalence of semi-stable points.*

Proof. □

0.2.3 Luca's Slice Theorem

Luna's slice theorem is a useful tool to study local property of the GIT quotient. suppose $G \times X \to X$ with G-linearized $L \in Pic(X)$

Definition:

(i) $x \in X$ is semistable w.r.t L if there is G-invariant section $s \in H^0(X, L)^G$ st: $s(x) \neq 0$ and

(ii) $x \in X$ is stable w.r.t L if

Theorem 0.2.5.

Proof. □

Theorem 0.2.6. 1. *If X is irreducible, then so is $X/\!\!/_L G$.*

2. *If X is normal, then so is $X/\!\!/_L G$.*

Proof. □

0.2.4 Hilbert-Mumford Criteria

Theorem 0.2.7. *(Hilbert-Mumford)*

(i) $p \in X$ *is semistable* $\Leftrightarrow$ *weight* $\mu(p, \lambda) \geq 0$ *for any* 1*-PS*λ

(ii) $p \in X$ *is polystable* $\Leftrightarrow$ *weight* $\mu(p, \lambda) > 0$ *for such* 1*-PS* λ *as lim*

(iii) $p \in X$ *is stable* $\Leftrightarrow$ *weight* $\mu(p, \lambda) > 0$ *for any* $1 - PS\ \lambda$

Proof. □

Picard group of GIT quotient
Let

$$\pi : \ X^{ss} \to X^{ss}/G = X//G$$

be guotient map. In **??**, Thomas Nevins give a necessary and sufficient conditions for descent problem of coherent sheaves and complex from X^{ss} to $X//G$.

Theorem 0.2.8. $X :=$ *scheme of locally finite type* $/\overline{char(k)} = 0$*.* $\mathcal{F} \in Coh(X)^G$*, then* $\mathcal{F}$ *descents to* $X//G$ *iff for* $\forall x \in X$ *closed with* $G \cdot x = \overline{G \cdot x}$*, the* $\mathcal{O}_x$*-module* $\mathcal{F} \otimes \mathcal{O}_X/\mathfrak{m}_x$ *and* $Tor_1^{\mathcal{O}_x}(\mathcal{F}, \mathcal{O}_X/\mathfrak{m}_x)$ *are generated by*

If $k = \overline{k}$*, then it is equivalent to require the action*

$$Stab_G(x) \curvearrowright \mathcal{F} \otimes \mathcal{O}_X/\mathfrak{m}_x,\ Tor_1^{\mathcal{O}_x}(\mathcal{F}, \mathcal{O}_X/\mathfrak{m}_x)$$

is trivial.

Proof. □

In particular, one has (see **??**)

$$0 \to Pic(X//G) \xrightarrow{\pi^*} Pic(X)^G \to\to 0$$

A Baby example–$|\mathcal{O}_{\mathbb{P}^2}(3)|//PGL(3) \cong M_1^{BB} \cong \mathbb{P}^1$:

0.2.5 Application 1: construction of coarse moduli space $\mathcal{M}_{g,n}$

0.2.6 Application 2: construction of coarse moduli $\mathcal{K}_d$ of K3

0.2.7 Application 3: construction of coarse moduli of vector bundles over curves

Theorem 0.2.9.

Proof. □

0.3 Some key techniques to detect GIT Stability

In general,the advantage of GIT's approach to moduli is that you can show the projectivity and compactify the moduli space very easily. But the disadvantage is very obvious too: very difficult to get detailed analysis the stability m.

0.4 moment maps and GIT

0.4.1 moment maps

Definition 0.4.1. *Let (X, ω) be a symplectic manifold with a compact Lie group action $G \curvearrowright X$. the moment map*

$$\mu : X \to \mathfrak{g}^* \cong \mathbb{R}^{\dim G}$$

defined by

$$d\langle \mu(x), \varsigma \rangle = \omega(V_\varsigma,)$$

where $\varsigma \in \mathfrak{g}$ and V_ς is the vector field generated by the flow $\varphi_t := exp(t\varsigma) \in Sym(X, \omega),\ t \in \mathbb{R}$, ie,

$$\frac{d\varphi_t}{dt}(x) = V_\varsigma(\varphi_t)$$

$$\varphi_0 = id$$

Here, the dual pairs $< \mu(x), \xi >$ is induced by killing form on $\mathfrak{g}$

Example 0.4.2. $(X, \omega) = (\mathbb{C}^n, \frac{i}{2} \sum dz_i d\overline{z}_i)$ *and $G = U(1)$ the natural action is given by multiplication.Then*

$$\mathfrak{g} = i\mathbb{R}$$

$$\varphi_t : (z_1, ..., z_n) \mapsto (e^{it\theta} z_1, ...e^{it\theta} z_n), i\theta \in \mathfrak{g}$$

$$\frac{d\varphi_t}{dt} = \sum \theta e^{it\theta} z_1 \frac{\partial}{\partial z_l} - \theta e^{-it\theta} z_1 \frac{\partial}{\partial \overline{z}_l}$$

$$V_\varsigma = \sum i\theta z_l \frac{\partial}{\partial z_l} - i\theta \overline{z}_l \frac{\partial}{\partial \overline{z}_l}$$

$$\omega(V_\varsigma,) = i\theta \sum (z_j d\overline{z_j} + \overline{z_j} dz_j)$$

Thus, the moment map is

$$\mu(z)(\theta) = \theta |z|^2, \quad \theta \in \mathbb{R}$$

***Remark**: consider symplectic quotient, easy to see*

$$\mu^{-1}(1)/G = \mathbb{P}^{n-1}$$

This coincides with quotient in algebraic geometry.

Example 0.4.3. $(X, \omega) = (\mathbb{P}^n, \sqrt{-1}\partial\overline{\partial} log(\sum z_i \overline{z}_i)$ *and* $G = U(n+1)$ *acts* $\mathbb{P}^n$ *naturally. Then the moment map is given by*

$$\mu : \mathbb{P}^n \to \mathfrak{u}(n+1)$$

$$z \mapsto \mu(z)(A) := \frac{z \cdot A \cdot z^\perp}{\|z\|^2} \tag{0.4.1}$$

As a consequence, for smooth $X \subseteq \mathbb{P}^n$ *with action* $G \subseteq U(n+1)$ *, the moment map is*

$$\mu : X \to \mathfrak{g}^*$$

$$z \mapsto \mu(z)(A) := \frac{z \cdot A \cdot z^\perp}{\|z\|^2} \; for \; A \in G \tag{0.4.2}$$

Example 0.4.4. $(X, \omega) = (\mathbb{C}^n, \frac{i}{2} \sum dz_i d\overline{z_i})$ *and* $G = U(n)$ *acts* $\mathbb{C}^n$ *naturally. Each element* $A \in G$ *can be diagonalized. assume*

$$A = diag(\sqrt{-1}\lambda_1, ...\sqrt{-1}\lambda_n), \; \lambda_i \in \mathbb{R}, \; \sum \lambda_i = 0.$$

Then the moment map is

$$z \mapsto \mu(z)(A) :=$$

Example 0.4.5. *This example is taken from HuYi's note* **??**.
consider $\mathbb{C}^* \curvearrowright \mathbb{P}^3$ *by* $\lambda \cdot [x, y, z, w] := [\lambda x, \lambda y, \lambda^{-1} z, w]$.*Then moment map*

$$\mu([x, y, z, w]) = \frac{|x|^2 + |y|^2 - |z|^2}{|x|^2 + |y|^2 + |z|^2 + |w|^2}.$$

It has critical values $\{-1, 0, 1\}$. *There is a wall-crossing phenomenon:*

0.4.2 Marsden-Weinstein symplectic reduction

Let compact Lie group $K \curvearrowright (X, \omega)$ be a symplectic action, ie, $K \leq Sym(X, \omega)$ and

$$\mu : \; X \to \mathfrak{k}^*$$

is the associated moment map.

Theorem 0.4.6. *If* $v \in \mathfrak{k}^*$ *is a regular value of* μ *and* $K \curvearrowright \mu^{-1}(K \cdot v)$ *is free action, then the quotient space* $\mu^{-1}(K \cdot v)/K$ *inherits symplectic structure from* (X, ω).

Proof. □

0.4.3 Kempf-Ness theorem

Theorem 0.4.7. *Let $X \subset \mathbb{CP}^n$ be projective manifold with reductive Lie group action $G \subset GL(n+1, \mathbb{C}^n)$, then $p \in X$ is poly-stable $\Leftrightarrow$ orbit $G \cdot p \bigcap \mu^{-1}(0) \neq \phi$. Moreover if p is polystable, then*

$$\#G \cdot p \bigcap \mu^{-1}(0) = 1$$

Proof. Suppose G is complexification of a compact subgroup $K \subset GL(n+1, \mathbb{C})$, fixed a $v_0 \in \mathbb{C}^n$ then consider

$$G \to \mathbb{R}$$

$$g \mapsto \mid g \cdot v_0 \mid$$

Note that

$$\mid g \cdot v_0 \mid = \mid k \cdot g \cdot v_0 \mid$$

for each $k \in K$, then it induces

$$G/K \to \mathbb{R}$$

and G/K is a homogenous space admit nonnegative curvature.
We claim: the function obtain its minimum iff v_0 is stable ☐

Hyperkahler Reduction

Now Assume (X, i, J, K) is a HK with $dim_{\mathbb{C}}(X) = 2n$ and $\omega_I, \omega_J, \omega_K \in H^2(X, \mathbb{R})$ kahler forms w.r.t metric $g_I, g_J, , g_K$. If G is a Lie group acting $(X, \omega_I, \omega_J, \omega_K)$, then we have 3 moment maps

$$\mu = (\mu_I, \mu_J, \mu_K): \ X \to \mathfrak{g}^* \oplus \mathfrak{g}^* \oplus \mathfrak{g}^*$$

Theorem 0.4.8.

Proof. ☐

0.4.4 Cohomology ring of Kirwan maps and Nonabelian localization

Assume $0 \in \mathfrak{g}^*$ is a regular value for compact Lie group G action on (X, ω).

Then G-embedding $(\mu^{-1}(0), G) \hookrightarrow (X, G)$ induces so-called Kirwan maps

$$k: \ H^*_G(X) \to H^*_G(\mu^{-1}(0)) = H^*(X//G) = H^*(X^s/G) \tag{0.4.3}$$

Theorem 0.4.9. *(Kirwan, Jeffrey-Kirwan, see [9])*

1. *Kirwan map k is surjective.*
2. *The kernel $ker(k)$ of k is the ideal generated by*

$$\{\alpha: \ \alpha|_F = 0, \ < \mu(F), \xi > 0, \ F \subset X^G \ \}_{\xi \in \mathfrak{g}^*} \tag{0.4.4}$$

Proof. ☐

$X = X^s \sqcup X^{ns}$ and $X^G \subset X^{ns}$

0.5 VGIT and wall-crossing

0.5.1 GIT approach to factorization of birational map

$Amp^G(X) :=$ space of G-invariant ample line bundles on $X \in Sm.Proj(k)$.

$Amp^G(X)$ is a rational polyhedral convex open cone with chamber decomposition

$$Amp^G(X) - \bigsqcup_i W_i = \bigsqcup_j \mathcal{C}_j$$

Theorem 0.5.1. *(M.Thaddeus [12], Dolgachev-Hu [2])*

1. *chamber decomposition: there are finitely many chambers*
2. *Wall-crossing induces natural flip: let C_+, C_- be a pair of adjacet chambers w.r.t wall W, then*
$$X^s = X^s_+ \cap X^s_- \subset X^{ss}_+ \cap X^{ss}_- \subset X^{ss}$$
which induces a flip
$$\begin{tikzcd} X^{ss}_+/\!\!/G \arrow[rr, dashed] \arrow[dr, "f^+"] & & X^{ss}_-/\!\!/G \arrow[dl, "f^-"] \\ & X^{ss}/\!\!/G & \end{tikzcd}$$
3. $E^+ := X^{ss}_+ - X^{ss}_-/G,\ E^- := X^{ss}_- - X^{ss}_+/G,\ Z := X^{ss} - X^s_+ \cap X^s_-$*, then the flip is*
$$\begin{tikzcd} E^+ \arrow[dr, "W\mathbb{P}^{d_+}-bundle"] & & E^+ \arrow[dl, "W\mathbb{P}^{d_-}-bundle"] \\ & Z & \end{tikzcd}$$
with dimension formula
$$d_+ d_+ + 1 = codim Z$$

Proof. □

By density of $\mathbb{Q}$ in $\mathbb{R}$, HM-index functions exsention to

$$\begin{aligned} \mu:\ & X \times \chi(G) \times Amp^G(X)_{\mathbb{R}} \to \mathbb{R} \\ & (x, \lambda, L) \mapsto \mu(x, \lambda, L) \end{aligned} \tag{0.5.1}$$

then the Hilbert-Mumford criterion shows

$$\begin{aligned} X^{ss}(L) &:= p_x(\mu^{-1}([0,+\infty)) \cap X \times \chi(G) \times \{L\}), \\ X^{s}(L) &:= p_x(\mu^{-1}((0,+\infty) \cap X \times \chi(G) \times \{L\}) \end{aligned} \tag{0.5.2}$$

where $p_x : \ X \times \chi(G) \times Amp^G(X)_{\mathbb{R}} \to X$ is natural projection.

Chamber Decomposition of Lie algebra $\mathfrak{t}$ of maximal torus T

Mori's dream space: Birational geometry of quotient space

Definition 0.5.2. *(see [8]) $X \in Proj.Var(k)$ Q-factorial and normal. If*

1. $Pic(X)_{\mathbb{Q}} = NS(X)$.
2. $Nef(X)$ *is affine hull of finitely many semi-ample line bundles.*
3. $Mov(X) = \bigcup_{f:X \dashrightarrow X_i} f_i^*(Nef(X_i))$ *the union is finite collection of small $\mathbb{Q}$-modification (ie, isomorphism in codimension 1).*

Known Examples

0.6 Non reductive GIT

Recently, Doran-Kirwan [4] [10] develop

0.7 Derived category of GIT

we review the work of [5], [1], [6]. These works relate derived geometry of X to its reducive GIT quotient $X/\!\!/G$.

Question 0.7.1. *Can we apply these description to Joyce-Song's counting thoery or Kontsevich-Soibelman's motivic DT ?*

A key observation may be that VGIT will naturally induce WSP of quotients, these birational maps are controlled by the GIT stability.

0.8 stability in algebraic geometry

A Central Problem in Kahler geometry: Given kahler class $\Omega \in Kah(X) \subset H^{1,1}(X)$ for a kahler manifold X, find the optimal metric g in Ω:

$$\sum \in \Omega$$

0.8.1 K-stability

K-stability originates in the KE problem in kahler geometry which is motivated by the work of Kobayashi-Hitchin, Yau-Ulnenbeack, Donaldson's work on the relation of stability of holomorphic bundles and curvature. when turn to cotangent bundle, it's KE problem(more broadly, CSCK). Due to Yau, Aubin-Yau, Tian, Futaki, Donaldson.

Futaki invatiant: obstruction to KE metric on Fano manifold

In history, the first obstruction to KE-metric is Mas 's $Aut(X)$, which says it's reductive if X admits KE. then in 1985, A.Futaki gives another one. Here we follow.

Set

$$\mathfrak{h} := \{\partial f \in \Gamma(TX) : f \in \mathcal{C}^\infty(X, \mathbb{C})\}$$

where $\partial f := g^{i\bar{j}} \frac{\partial f}{\partial \bar{z}_j} \frac{\partial}{\partial z_i}$

Calabi functional

$$\mathcal{F} : \mathfrak{h} \to \mathbb{C}$$
$$f \mapsto \int_X (S - \widehat{S}) \cdot f \cdot \omega^n$$

where scalar curvature of kahler metric and average scalar curvature

$$S := g^{i\bar{j}} R_{i\bar{j}},\ \widehat{S} := \frac{\int_X S \cdot \omega^n}{\int_X \omega^n}$$

It's easy to see if X admits CSCK, then

$$\mathcal{F} \equiv 0$$

Note that KE is just a special CSCK

Computation of variantional equation for calabi functional: Assume ω is the reference kahler metric and take a pertuation

$$\omega_t := \omega + t\sqrt{-1}\partial\bar{\partial}\psi,\ t \in \mathbb{R}$$

then

$$\begin{aligned} \frac{d\omega_t^n}{dt}|_{t=0} &= \frac{d}{dt}(\omega + t\sqrt{-1}\partial\bar{\partial}\psi)^n \\ &= \frac{d}{dt}(\sum_l \binom{n}{l} t^l \cdot \omega^{n-l} \wedge (\sqrt{-1}\partial\bar{\partial}\psi)^l) \\ &= n \cdot \sqrt{-1}\partial\bar{\partial}\psi \wedge \omega^{n-1} \end{aligned} \tag{0.8.1}$$

By choosing a normal coordinate st: $\omega = \sqrt{-1}(dz_1 d\bar{z}_1 + \cdots + dz_n d\bar{z}_n)$, then
Thus,

$$\frac{d\omega_t^n}{dt}|_{t=0} = n \cdot \sqrt{-1}\partial\bar{\partial}\psi \wedge \omega^{n-1} = \Delta\psi \cdot \omega^n \tag{0.8.2}$$

where the Laplacian operator Δ is w.r.t metric ω.

Let

$$\mathcal{H}_\omega := \{ \phi \in C^\infty(X) : \omega + \sqrt{-}\partial\bar{\partial}\phi > 0 \}$$

be the space of kahler metrics and ψ_0 be the kahler potential, ie,

$$\omega = \sqrt{-}\partial\bar{\partial}\psi_0$$

with $V := \int_X \omega^n = \int_X \omega_\phi^n = L^n$.
Define

$$E(\phi) := \frac{1}{(n+1)V} \int_X \sum_{i=0}^{n} \int_X (dd^c(\phi + \psi_0)^i \wedge (dd^c\psi_0)^{n-i} \tag{0.8.3}$$

Fact:

$$I(\phi) = \tag{0.8.4}$$

Fact:
Mabul functional:
Ding functional:
Conjecture: Yau-Tian-Donaldson
Let (X, L) be a polarised Kahler manifold (variety), there is a CSCK metric g in $c_1(L)$ iff (X, L) is K-polystable.

0.8.2 Algebraic geometric formulation of K-stability:

Donaldson-Futaki invariant: Algebraic geometry enters

Definition 0.8.1. *(Test-configuration) Let (X, L) be a polarized manifold with L ample. A Test-configuration for of exponent $r > 0$ is an embedding $X \hookrightarrow \mathbb{CP}^{N_r}$ by $L^{\otimes r}$ with 1-PS $\lambda : \mathbb{C}^* \to GL(N_r + 1, \mathbb{C})$ where $N_r := dimH^0(X, L^{\otimes r})$*

Example 0.8.2. *(Trivial TC)*

Example 0.8.3. *(specail TC) (X, L) is STC if (X, X_0) is plt, equivalently, X_0 is normal.*

Now assume X is fano and $L = -K_X$ ample. Denote the space of TC of exponent r

$$\mathtt{TC}_r(X) := \{ (\mathcal{X}, \mathcal{L}) \to \mathbb{C} : \ \mathbb{C}^* - equivariant, \ (\mathcal{X}_t, \mathcal{L}_t) \sim (X, -rK_X) \}$$

$$\mathtt{TC} := \bigcup_{r \geq 1} \mathtt{TC}_r, \ \ \mathtt{STC} \subset \mathtt{TC}$$

The space of 1-PS

$$Hom(\mathbb{C}^*, SL(H^0(X, -rK_X)) := \{ \mathbb{C}^* \ \xrightarrow{\lambda} \ SL(H^0(X, -rK_X) \}$$

Fact: There is 1-1 between $Hom(\mathbb{C}^*, SL(H^0(X, -rK_X))$ and $\mathtt{TC}_r(X)$:
Given $\lambda \in Hom(\mathbb{C}^*, SL(H^0(X, -rK_X))$, then

$$\lambda : \ \mathbb{C}^* \to Hil_p(\mathbb{P}^{N_r}), \ \ t \ \mapsto \ \lambda(t)[X]$$

by adding $[X_0] := \lim_{t \to 0} \lambda(t)[X]$, the limit exists and is unique since hilbert scheme is proper and separated, so pullback from universal family $\mathcal{X} \subset$

Tian's analytical definition

Donaldson's algebraic definition

Let $(\mathcal{X}, \mathcal{L}) \to \mathbb{C} \in \mathtt{TC}_r(X)$ be a TC for X.

$$N_r(m) = \dim H^0(X, -K_m) = a_0 \cdot m^n + a_1 \cdot m^{n-1} + o(m^{n-1})$$

$$W_r(m) := Totalwt \ (\mathbb{C}^* \curvearrowright H^0(\mathcal{X}, \mathcal{L})) = b_0 \cdot m^{n+1} + b_1 \cdot m^n + o(m^n)$$

then Donaldson defined Donaldson-Futaki invariant in [3] as

$$DF(\mathcal{X}, \mathcal{L}) := \frac{a_0 \cdot b_1 - b_0 \cdot a_1}{a_0^2}$$

Comparing w.r.t GIT, this is analogue of Hilbert-Mumford index.
By Equivarant Riemann-Roch,

Proposition 0.8.4. *(Donaldson, 02 [3])*

K-stability and Hilbert stability (or Chow stability)

$$\texttt{Fano}_{V,n}^{k-poly} := \{\ X:\ \mathbb{Q} - Fano\ n - fold,\ k - polystable,\ (-K_X)^n = V\ \ \}$$

Relying on solution of BAB conjecture due to Birkar, Jiang show that $\forall\ X \in$ Fano$_{V,n}^{k-poly}$, there is a universal integer $r_0 = r_0(n, V) \in \mathbb{N}$ st:

$$|-rK_X|:\ X \hookrightarrow \mathbb{P}H^0(X, -rK_X)$$

One may use $Hil_p(\mathbb{P}^{N_r})$ the Hilbert scheme of closed subschemes in $\mathbb{P}^{N_r}$ (or chow variety), where $p(m) := \chi(-mrK_X))$

Intersection formula for DF invariant

Theorem 0.8.5. *(Xiaowei Wang, Y.Odaka)*

Proof. By gluing

$$\begin{tikzcd} (\mathcal{X}, \mathcal{L}) \arrow[d, "\mathbb{C}^*-equi"] & X \times \mathbb{C}^* \arrow[l, hook'] \arrow[r, hook] \arrow[d] & X \times \mathbb{P}^1 - \{0\} \arrow[d] \\ \mathbb{C} & \mathbb{C}^* \arrow[l, hook'] \arrow[r, hook] & \mathbb{P}^1 - \{0\} \end{tikzcd}$$

□

Special TC and Li-Xu's work

By the work of Li-Xu, (see [11],It's enough to only consider special test configurations.

Lemma 0.8.6. *Suppose X is $\mathbb{Q}$ Fano and Let $(\mathcal{X}, \mathcal{L}) \to \mathbb{C}$ be a TC for $(X, -rK_X)$*

Theorem 0.8.7.

Idea of their proof is follows:
step1: By recent results in MMP(especially BCHM), they can modify the TC to a 'good' degeneration, ie, a special TC.
step2: Using intersection formula for DF invariant to show along the modification, the DF invariant decrease.

Odaka-Xu, Odaka's work on lc modification

Theorem 0.8.8. *(Odaka)*
Let (X, L) be K-semistble and X is normal $\mathbb{Q}$-Gorenstein, then X is lc. Moreover, if X is Fano, then X is klt.

Proof. □

Tian's CM stability

Summary of Known Numerical condition to characterise K-stability

Theorem 0.8.9. *(Ruadhai.Dervan, Characterizations by alpha invariant)*
Let (X, L) be a polarized $\mathbb{Q}$-Gorenstein lc variety with

- $\alpha(X, L) \geq \frac{n}{n+1}\mu(X, L)$ *where slope*

$$\mu(X, L) := \frac{-K_X.L^{n-1}}{L^n} = \frac{\int_X c_1(X)c_1(L)^{n-1}}{c_1(L)^n}$$

- $-K_X \geq \frac{n}{n+1}\mu(X, L)L$.

Then (X, L) is K-stable.

Proof. □

Definition 0.8.10. *(Log canonical threshold)*
Let (X, D) be

The progress on construction moduli space of K-stable Fano varieties (Taken from a lecture of prof Xu):

	Smoothable (analytic)	$\mathbb{Q}$-Fano
Boundedness	CDS, Tian	Jiang (BAB)
openness	CDS, Tian	??
completeness	CDS, Tian	??
separatedness	LWX,SSY	?
projectivity	partcialy LWX	?

Some questions:

1. To verify K-stability of

- 3-fold of genus$= 12$
- cubic 4-fold
- all smooth hypersurface

2. Normalized volume of singularities:

0.8.3 Valuations methods

Set

$$Val_{X,x} : \{ \ v : K = \mathbb{C}(X) \to \mathbb{R} \ \in Val_X : \ \mathcal{O}_v \subset \mathcal{O}_{X,x} \}$$

A biational model $E \hookrightarrow Y \to X$ will give a natural valuation (divisorial valuation)

$$ord_E : \ \mathbb{C}(Y) = \mathbb{C}(X) \ \to \ \mathbb{R}, \ \ f \ \mapsto$$

We denote

$$DVal_X \ \subset Val_X, \ \ DVal_{X,x} \ \subset Val_{X,x}$$

quasimomonial valuation Let $E = E_1 + \cdots + E_r$ be snc w.r.t $\mu : Y \to X$ with $\cap E_i \neq \phi$ and locally at genric point $\eta \in C \subset \cap E_i$, for $\alpha ;= (\alpha_1, \cdots, \alpha_r) \in \mathbb{R}^r_{>0}$, define

$$v_\alpha(f \min_{c_\beta(\eta)\neq 0} \{ \sum \alpha_i \ \}$$

$$QM_X \ \subset DVal_X \subset Val_X$$

log discrapence function

$$\begin{aligned} A_X : \ Val_{X,x} \to \mathbb{R} \\ A_{(X,D)} : \ Val_{X,x} \to \mathbb{R} \end{aligned} \tag{0.8.5}$$

$$\begin{aligned} Vol_X(-K_X - t\cdot v) &:= \lim_{m\to\infty} \frac{\dim\{ \ s \in H^0(-mK_X) : v(s) \geq m\cdot t \ \}}{m^n/n!} \\ &= \lim_{m\to\infty} \frac{h^0(\mathcal{O}_X(-mK_X)\otimes \mathfrak{a}_{tm})}{m^n/n!} \\ T_X(v) &:= \sup\{\lambda > 0 : \ Vol_X(-K_X - t\cdot v) > 0\} \end{aligned} \tag{0.8.6}$$

where $\mathfrak{a}_{tm}(v) := \{f \in \mathcal{O}_X : v(f) \geq m\cdot t\}$ is the ideal sheaf associated to v.

Theorem 0.8.11. *(Li-Liu,Li,Fujita) Local to global volume comparasion.*
(X, D) is k-s.s, then for any $v \in Val_x$,

$$\hat{Vol}(v) \geq (K_X + D)^n \cdot (\frac{n}{n+1})^n$$

Proof. A simple case: $x \in X$ is a smooth point, take $\mu : Y \to X$ blowup at x, then

$$A_X(E) = 1 + coff_E(K_Y - \mu^* K_X) = 1 + (n-1) = n$$

then by Fujita-Li's valuation criterion for K-s.s, $\beta_X(E) \geq 0$ implies

$$\begin{aligned} A_X(E) - S_X(E) &= n - \frac{}{(-K_X)^n} \\ &\geq n - \frac{}{(-K_X)^n} \end{aligned} \tag{0.8.7}$$

□

0.8.4 Filtration methods

$R := \bigoplus_m R_m := \bigoplus_m H^0(X, -m \cdot \ rK_X)$ or its log version

$$R := \bigoplus_m R_m := \bigoplus_m H^0(X, -m \cdot r(K_X + D))$$

where r is the cartier index.

Definition 0.8.12. $\{\mathcal{F}^t R\}_{t\in\mathbb{R}}$ *is called* *Filtration* *on* R *if*

multiplicativity $\mathcal{F}_v^a R_m \cdot \mathcal{F}_v^b R_n \subset \mathcal{F}_v^{a+b} R_{m+n}$ *for any* $a, b \in \mathbb{R}, \ m, n \in \mathbb{N}$.

Bounded

there is a natural map from valuations to filtration

$$Val_X \to Fil(R), \ \ v \ \mapsto \ \{\mathcal{F}_v^t\}$$

where $\mathcal{F}_v^t R_m := \{ \ s \in H^0(X, -m \cdot rK_X) : \ v(f) \geq t \ \}$.

0.8.5 Chow-stability

Definition 0.8.13. *A normal variety $X \subset \mathbb{P}^N$ is called chow stable(semi-stable) if its chow form (chow point) is stable(semi-stable) in the sense of GIT $SL(N+1, \mathbb{C}) \curvearrowright Chow$*

Definition 0.8.14. *A polarized variety (X, L) is called asymptotic chow stable(semi-stable) if $\varphi_m(X) \subset \mathbb{P}^{N_m}$ is chow stable(semi-stable) for $m \gg 0$. Here φ_m is the embedding giving by $|L^{\otimes m}|$*

Proposition 0.8.15. *asymptotical chow stable $\Rightarrow$ asymptotical Hilbert stable $\Rightarrow$ asymptotical Hilbert semi-stable $\Rightarrow$ asymptotical Chow semi-stable $\Rightarrow$ K-smeistable*

Proof. □

Examples

$$\begin{aligned} Hil_p(\mathbb{P}^N) &\hookrightarrow Grss(p(m),) \\ [Z] &\mapsto [H^0()] \end{aligned} \tag{0.8.8}$$

0.8.6 Bridgeland-stability

Let X be a n dimensional smooth projective variety and $\mathcal{D}^b(X)$ be the bounded derived category of $Coh(X)$.

Definition 0.8.16. *A stability condition $o = (Z, P)$ on $\mathcal{D}$ consists of*

1. *central change $Z : K(\mathcal{D}) \to \mathbb{C}$*
2. *a family of full additive subcategories $\{\mathcal{P}(\phi) \subset \mathcal{D}\}_\phi$*

such that

- $0 \neq E \in \mathcal{P}(\phi)$,*then* $\exists\ m(E) > 0$ *st:*$Z(E) = m(E)e^{i\pi\phi}$
- $\mathcal{P}(\phi)[1] = \mathcal{P}(\phi + 1)$
- $\phi_1 > \phi_2$ *and* $E_j \in \mathcal{P}(\phi_j)$,*then* $Hom_{\mathcal{D}}(E_1, E_2) = 0$
- $0 \neq E \in \mathcal{P}(\phi)$,*then* $\exists$
$$\phi_1 > \phi_2 > ..\phi_n \in \mathbb{R}$$
and triangles
$$\begin{tikzcd} 0 = E_0 \arrow[rr] \arrow[dr] & & E_1 \arrow[dl] \\ & A_1 & \end{tikzcd}$$

Inspired work of M.Doalglas on stability in string theory, T.Bridgeland introduce such stability and define the space of stability condition

$$stab(\mathcal{D}) := \{\sigma \ stability \ condition\}$$

with a metric

$$d(\sigma_1, \sigma_2) := \sup_{0 \neq E \in \mathcal{D}} \{\mid \phi^-_{\sigma_1}(E) - \phi^-_{\sigma_2}(E) \mid, \ \mid \phi^+_{\sigma_1}(E) - \phi^+_{\sigma_2}(E) \mid, \ \mid \lg \frac{m_{\sigma_2}(E)}{m_{\sigma_1}(E)} \mid\}$$

where $\phi^-_\sigma(E) := \phi_n$, $\phi^+_\sigma(E) := \phi_1$ as given in definition. Then prove the following fundamental theorem **??**

Theorem 0.8.17. *(T.Bridgeland,06)*
$stab(\mathcal{D})$ is a complex manifold.

Proof. □

0.9 GW on GIT quotient: quasimaps

0.10 Appedix: Hilbrt scheme, Chow variety and Quot schemes

Grassimian $G(n, m)$ is a toy model. It parameterizes all n- dimensional subspace of $\mathbb{C}^m$, or equivalently, all (n-1)-dimensional projective subspace in $\mathbb{P}^{m-1}$

0.10.1 Construction of Hilbert schemes

Hilbert scheme is a moduli space parameterizing all closed subschemes with given Hilbert polynomial in a given projective space.

Lemma 0.10.1. *(Uniform lemma)*
Given polynomial P, if $Z \subset \mathbb{P}^N$ is a closed subscheme with Hilbert polynomial P, Let be $\mathcal{I}$ its ideal sheaf. then $\exists$ integer $m = m(P)$ st: for $n \geq m$

(i) $h^i(\mathbb{P}^N, \mathcal{I}(n)) = 0$ *for* $i > 0$

(ii) $\mathcal{I}(n)$ *is generated by global sections.*

(iii) $H^0(\mathbb{P}^N, \mathcal{O}(n)) \twoheadrightarrow H^0(X, \mathcal{O}_X(n))$

Proof. The proof follows induction on N.
Recall by cohomological definition of Hilbert polynomial,

$$P(n) = \chi(\mathcal{O}_X(n)) = dimH^0(X, \mathcal{O}_X(n)) \ for \ n \gg 0$$

since $\mathcal{O}_X(1)$ is very ample.
$N = 0$, it's trivial.
Now suppose it holds for $< N$
Take H a hyperplane of $\mathbb{P}^N$ st:each component of X $\not\subseteq H$. Set $\mathcal{J} := \mathcal{I} \otimes \mathcal{O}_H$ By tensoring $\mathcal{O}_H$ with

$$0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_X \to 0$$

we have

$$\mathcal{J} \hookrightarrow \mathcal{O}_H$$

By induction hyperthesis, $\exists \ n_0 = n_0(N, \mathcal{J})$ st:

$\square$

Let $H_{P,N} := \{Z \subset \mathbb{P}^N \ with \ P_Z = P\}/_{\simeq}$.Then choose a uniform integer n in lemma0.10.1, $H_{P,N}$ is embedded into Grassimian:

$$H_{P,N} \to G(P(n), \binom{N+n}{n}) = G(\binom{N+n}{n} - P(n), \binom{N+n}{n})$$
$$via \ Z \mapsto \{H^0(\mathcal{I}(n)) \subset H^0(\mathcal{O}_{\mathbb{P}^N}(n))\}$$

By Plucker embedding,

$$G(n, m) \hookrightarrow \mathbb{P}^A = \mathbb{P} \wedge^n \mathbb{C}^m$$

$$span(v_1, ..v_n) \mapsto v_1 \wedge ... \wedge v_n$$

Interpresentated as Moduli problem: consider functor

$$\mathcal{H}_{P,N} : sch(S) \to sets$$
$$T \mapsto \{\mathcal{X} \to T flat\ proper\ fiber\ isomorphic\ to$$
$$closed\ subscheme\ in\ \mathbb{P}^N with\ Hilbert\ polynomial\ P\}$$

Theorem 0.10.2. *$\mathcal{H}_{P,N}$ is represented by $H_{P,N}$ with a universal family $\mathcal{U} \to H_{P,N}$*

Proof. idea of construction:

step1: Grassmannian can be represented. Given a Noetherian scheme and a vector bundle E on S,$rk(E) > r$,consider

$$\mathcal{G}r : sch(S) \to sets$$
$$T \mapsto \{subbundles\ of\ T \times_S E\ with\ rank\ = r\}$$

let $t_1, ...t_n \in H^0(S, E)$ be global sections of E generating E.

For each$s \in S$,E_s is $k(s)$ vector space,we can associate grassmmian this define a scheme $G(r, E)$ over S More precisely,using plucker embedding

claim: $\mathcal{G}r$ is represented by $G(r, E)$. consider transformation

$$T :$$
$$a$$

step2: Hilbert functor is related to Grassmannian

□

Remark: In general, we can consider the Hilbert scheme parameterizing subschemes in a general projective scheme X over S.

Example 0.10.3. *The Hilbert scheme of hypersurface of degree d in $\mathbb{P}^n$*

$$Hilb_p(\mathbb{P}^n) = \mathbb{P}(H^0(\mathbb{P}^n, \mathcal{O}(d))$$

Example 0.10.4. *The Hilbert scheme of points on smooth curves C of length m.*

Example 0.10.5. *The Hilbert scheme of points $X^{[m]}$ on K3 X of length m.*

Theorem 0.10.6. *(Fogarty, A.Fujiki, A.Beaville,)*

$X^{[m]}$ is a compact smooth irreducible holomorphic variety of dimension $2m$ and its betti number

$$b_k(X^{[m]})$$

Proof. The Hilbert-chow morphism is just resolution of singularities

$$X^{[m]} \to X^{(m)}$$

Away from singular locus, there is a natural holomorphic symplectic $2-$forms on $X^{(m)}_{reg}$ by pullback of from □

Example 0.10.7. *The Hilbert scheme of points $Hil_m(\mathbb{C}^2)$ on affine plane of length m. Surprisingly, $Hil_m(\mathbb{C}^2)$ have very rich geometric structure to representation theory, see.*

Theorem 0.10.8. *Define*

$$\mathcal{M} := \{(A_1, A_2, v) \in End(\mathbb{C}^n)^2 \times \mathbb{C}^n : [A_1, A_2] = 0; stability\}//GL(n, \mathbb{C})$$

where stability means No $L \subsetneq \mathbb{C}^n$ st: $A_i(L) \subset L,\ v(L) \subset L$ and group action is conjugate, ie,

$$g \cdot (A_1, A_2, v) := (gA_1g^{-1}, gA_2g^{-1}, gv)$$

Then we have

$$\mathcal{M} \cong Hil_n(\mathbb{C}^2)$$

and it's a smooth projective variety of $dim = 2n$.

Proof.

$$X^{[m]} \to X^{(m)}$$

recall

□

Example 0.10.9. *The Nested Hilbert scheme of points on smooth surface X of length m.*

Example 0.10.10. *Mumford's nonreducedness example:*

Local structure of Hilbert scheme: tangent space

Basic tool of studying local structure of Hilbert scheme is deformation theory.Here only list the main result, for deformation theory, a nice reference is [7].

Theorem 0.10.11. *For $Y \subset X$ closed subset of a fixed projective scheme X and the Hilbert point corresponding to $[Y]$ is a smooth point, then the tangent space of $Hil_p(X)$ is given by*

$$T_{[Y]}Hil_p(X) \cong H^0(Y, N_{Y/X})$$

Proof. □

Theorem 0.10.12. *(Harshone)*
$H_{P,N}$ is connected

Proof. □

0.10.2 Construction of Chow Variety

Comparing with Hilbert scheme, Chow Varieties parameterize cycles of fixed codimension in an Variety.

• **Chow forms of** $X \subsetneqq \mathbb{P}^n$

Let $X \subsetneqq \mathbb{P}^n$ be an irreducible variety of dimension $= k$ and degree $= d$. Set

$$B(X) := \{(x, L) \in X \times Gr(n-k, n+1) : x \in L\}$$

$$Z(X) := \{L \in Gr(n-k, n+1) : L \cap X \neq \phi\}$$

Then there are natural projections:

$$\begin{tikzcd} B(X) \arrow[r, "p_2"] \arrow[d, "p_1"'] & Z(X) \\ X & \end{tikzcd}$$

Theorem 0.10.13. *$Z(X) \subseteq Gr(n-k, n+1)$ is an irreducible hypersurface of degree $=$ $deg(X) = d$*

Proof. note that the general fiber of p_1 :

$$p_1^{-1}(x) \approx Gr(n-k-1, n)$$

and by dimension counting

$$\dim(L) + \dim(X) - \dim(\mathbb{P}^n) = (n-k-1) + k - n = -1$$

So p_1 is a degree$=1$ map and thus a birational map, then run dimension counting,

$$\dim(B(X) = \dim(Z(X) = \dim(X) + \dim(Gr(n-k-1, n)$$

$$= k + (n-k-1)(k+1) = \dim(Gr(n-k, n+1) - 1$$

Fixed a general projective subspace M, N of $\dim = n-k-2, n-k$ respectively. Then

$$C(M, N) := \{L \in Gr(n-k, n+1) : M \subset L \subset N\}$$

gives a generic line in $Gr(n-k, n+1)$ thus, the degree of $Z(X)$ should be $deg(Z(X)) = \#C(M, N) \cap Z(X) = deg(X) = d$ □

Let Φ_X be defining equation of $Z(X)$, ie,

$$\Phi_X \in H^0(Gr(n-k, n+1), \mathcal{O}(d))$$

Using Plucker embedding,

Example 0.10.14. *The chow variety parameterizing* $deg = 1$ *effective cycles is just Grassimian*

$$Chow(\mathbb{P}^n, 1, k) = Gr(k+1, n+1)$$

Example 0.10.15. *The chow variety* $chow(1, 2, \mathbb{P}^3)$ *parameterizing* $deg = 2$ *effective* $1-$*cycles in* $\mathbb{P}^3$ *has two irreducible components* Ξ_1 *and* Ξ_2 *parameterizing planar quadratics and pairs of lines in* $\mathbb{P}^3$ *respectively, moreover* $\Xi_1 \cap \Xi_2$ *parameterizing coplanar two lines. In fact, a non-degenerate variety will follow* $deg \geq 1 + codim$

Example 0.10.16. *The chow variety* $chow(0, d, \mathbb{P}^n)$ *parameterizing* $deg = d$ *effective* $0-$*cycles in* $\mathbb{P}^n$ *is just symmetric product* $sym^d(\mathbb{P}^n)$ *of* $\mathbb{P}^n$. *In fact, it holds for arbitrary variety* X.

Example 0.10.17. *The chow variety* $chow(n-1, d, \mathbb{P}^n) \cong |\mathcal{O}_{\mathbb{P}^n}(d)|$ *parameterizing* $deg = d$ *effective* $n-1$*-cycles in* $\mathbb{P}^n$ *is same as the Hilbert scheme.*

Example 0.10.18. $chow(2, 2, \mathbb{P}^4)$

For a 2-cycle $S \in chow(2, 6, \mathbb{P}^4)$ *of degree* $= 6$

Let $\nu := \#$ *degenerate component of* S. *We denote* S_i *and* X_j *its degenerate component and non-degenerate component. then all the possibilities are*

1. $\nu = 0$, *then* $S = X_1 + X_2$ *two irreducible degree* $= 3$ *surfaces or* $S = X$ *an irreducible degree* $= 3$ *surface*
2. $\nu = 1$, *then* $S = S_1 + X$ *or* S *irreducible degree* $= 6$ *degenerate surface*
3. $\nu = 2$, *then* $S = S_1 + S_2$ *or* $S = S_1 + S_2 + X$
4. $\nu = 3$, *then*
5. $\nu = 4$, *then* S *consists of 3* $\mathbb{P}^2$ *and*
6. $\nu = 5$, *then* S *consists of 4* $\mathbb{P}^2$ *and* $V(l, q)$
7. $\nu = 6$, *then* S *consists of 6* $\mathbb{P}^2$

Theorem 0.10.19. *(F.Catanese, see* **??***) The Hilbert-Chow morphism*

$$\varphi : \ Hil_p(\mathbb{P}^n) \to chow(k, d, \mathbb{P}^n)$$

Let $Hil^0 \subset Hil_p(\mathbb{P}^n$ *be the open locus of smooth (resp.) irreducible subvariety, then the reduced part* Hil^0_{red} *is isomorphic (resp. homemorphic) to its image* $\varphi(Hil^0_{red})$.

Proof. □

Corollary 0.10.20. *The main irreducible component of* $chow(2, 6, \mathbb{P}^4)$ *parameterizes the complete intersection of type* $(2, 3)$ *and their degenerations.*

Proof. for a $X \in Hil$ complete intersection of type $(2,3)$, the normal bundle is $N = \mathcal{O}(2) \oplus \mathcal{O}(3)$, then by KV-vanishing and RR,

$$H^1(X,N) = 0, \;\; h^0(X,N) = 14 + 29 = 43$$

□

• **A Mumford's Criterion for Chow stability of projective variety $X \subset \mathbb{P}^n$.**

The compactification is a

Example

Let $X = V(Q) \subset \mathbb{P}^3$ be a smooth quadric. then the chow point $c_X \in Chow(2,1;\mathbb{P}^3)$ is chow stable:

Note that

$$X \cong \mathbb{P}^1 \times \mathbb{P}^1, \; [u,v] \times [z,w] \mapsto [uz, uw, vz, vw]$$

we can identify the globally sections $\mathcal{O}(m)$ of $X \times \mathbb{A}$ as

$$\bigoplus_{\lambda \geq 0} R_m t^\lambda = span\{ \; u^i v^j z^k w^l t^\lambda : \; i+j = k+l = m, \lambda \geq 0 \; \}$$

where $R = \oplus_{m \geq 0} R_m$ is coordinate ring of X. For 1-PS with weight $(\lambda_0, ..., \lambda_4)$, the ideal $I =< t^{\lambda_0} x_0, .., t^{\lambda_4} x_4 >$, thus

$$\dim(H^0(X \times \mathbb{A}^1, \mathcal{O}(m))/I^m)$$

$$= \dim span\{u^i v^j z^k w^l t^\lambda : \; i+j = k+l = m, \lambda < \lambda_0 a_0 + .. + \lambda_4 a_4\}$$

$$= \sum_{i=0}^{m} \sum_{a_0=0}^{i} \sum_{a_2=0}^{m-i} \lambda_0 a_0 + \lambda_1(i - a_0) + \lambda_2 a_2 + \lambda_3(m - i - a_2)$$

$$= e_\lambda(X)\frac{m^4}{4!} + O(m^3)$$

So,

$$e_\lambda(X) = \lambda_0 + ..\lambda_4 < \frac{1 + \dim X}{1+3} deg(X) \sum \lambda_i = 1.5(\lambda_0 + ..\lambda_4)$$

0.10.3 Construction of Quot schemes

Quot schemes is a building block for construction of moduli space of sheaves over a variety.

0.10.4 Hilber-Chow morphism

Given closed a closed subscheme $Z \subset X$

Theorem 0.10.21. *The Hilbert-Chow morphism*

$$hc: \ Hil(\mathbb{P}^n) \to Chow(\mathbb{P}^n)$$

is proper.

Proof. □

0.10.5 Some basic Properties of Hilbert scheme and chow variety

Acknoeledgement 1. *During writing the notes, Dr GuangSheng Yu gave me lots of help in latex*

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