Cohomological stabilization for moduli space of 1D sheaves on $\mathbb{P}^2$

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Background and motivation

Fixed $d \in \mathbb{Z}_{>0}$ and $\chi \in \mathbb{Z}$ such that $\gcd(d,\chi)=1$, denote by $M_{d,\chi}$ be the moduli space of stable pure one dimensional sheaves $F \in \mathrm{Coh}(\mathbb{P}^2)$ on $\mathbb{P}^2$ with $\mathrm{Supp}(F) \in |\mathcal{O}_{\mathbb{P}^2}(d)|$ and Euler characteristic number $\chi(F)=\chi$. Here stability is with respect to

$$\mu(F) = \frac{\chi}{c_1(F).\mathcal{O}(1)}.$$

$M_{d,\chi}$ is a smooth projective variety of dimension d^2+1. The study of cohomology $H^*(M_{d,\chi})$ is motivated by the following perspective.

- the perverse filtration on $H^*(M_{d,\chi})$ categorifies the refined BPS invariants of $Tot(\omega_{\mathbb{P}^2})$.
- the Betti numbers is related to relative Gromov–Witten invariants due to P. Bousseau.
- "P=C" conjecture as a compact analogue of "P=W" conjecture in non-abelian Hodge theory.
- Stabilization conjecture on moduli space of stable sheaves.

Throughout the cohomology is $\mathbb{Q}$-coefficient.

What is "P" ?

By taking the Fitting support, there is a proper morphism

$$h: M_{d,\chi} \to |\mathcal{O}_{\mathbb{P}^2}(d)| = \mathbb{P}^N.$$

Beilinson-Bernstein-Deligne-Graber's decomposition theorem implies that

$$Rh_*\mathbb{Q} \;\cong\; \bigoplus_i \mathcal{P}_i[-i] \;\in\; D^b_c(\mathbb{P}^N).$$

By taking cohomology we have perverse filtration

$$P_l H^i(M_{d,\chi}) := \mathrm{Im}\Big\{H^{i-N}(\mathbb{P}^N, \bigoplus_{i=0}^{l}\mathcal{P}_i[-i]) \to H^i(\mathbb{P}^N, Rh_*\mathbb{Q})\Big\}.$$

What is "C" ?

Let

$$\mathcal{F} \to M \times \mathbb{P}^2$$

be an universal sheaf. Denote $H \in H^2(\mathbb{P}^2)$ the line class and define the normalized tautological class

$$c_i(j) := \pi_{1*}(\pi_2^* H^j \cdot ch^{\alpha}_{i+1}(\mathcal{F}))$$

where π_l is the projection on the l-th component and $\alpha \in H^2(M\times\mathbb{P}^2)$ such that

$$c_1(0)=0, \quad c_1(1)=0.$$

Theorem (Beauville, Markman,Pi-Shen[4])

$c_i(j) \in H^{2(i+j-)}(M)$ is independent of choice of the universal sheaf and they generate the cohomology ring as $\mathbb{Q}$-algebra.

Then define Chern filtration by

$$C_l H^*(M_{d,\chi}) := \mathrm{Span}_{\mathbb{Q}}\{\prod_{i=1}^{s} c_{k_i}(n_i) \mid \sum_{i=1}^{s} k_i \le l\,\}.$$

P=C Conjecture:

Conjecture (Kononov-Pi-Shen[2], Kononov-Lim-Pi-Moreira[1])

The Perverse filtration and Chern filtration on $H^*(M_{d,\chi})$ coincide, i.e,

$$P_l H^*(M_{d,\chi}) = C_l H^*(M_{d,\chi}).$$

Main results

Theorem A (Pi-Shen-Si-Zhang 2024 [5])

The "$P=C$" holds for the cohomological degree $\le 2d-4$.

Theorem B (Pi-Shen-Si-Zhang 2024[5])

A refined cohomological stabilization formula: the BPS number

$$n^d_{i,j} := \dim P_i H^{i+j}(M_{d,\chi})/P_{i-1}H^{i+j}(M_{d,\chi}) = H(q,t)_{i,j}$$

for $i+j<2d-4$ where

$$H(q,t) := \prod_{i\ge 0}\frac{1}{(1-(qt)^i q^2)(1-(qt)^{i+2})(1-(qt)^i t^2)}.$$

Idea of proof

- Maulik-Yun and Migliorini–Shende's full support result implies that
$$H^m(\mathcal{C}_U^{[n]}) \cong \bigoplus_{i+j\le n, j>0} \mathrm{Gr}_i^P H^{m-2j}(J_U)$$
where
$$\mathcal{C}_U^{[n]} \to U$$
is the relative Hilbert scheme of points and
$$J_U \to U$$
is the relative compactified Jacobian fibration over the integral locus $U \subset |\mathcal{O}_{\mathbb{P}^2}(d)|$.
- The geometry of relative Hilbert scheme of points: the natural morphism
$$\mathcal{C}_U^{[n]} \to (\mathbb{P}^2)^{[n]}$$
has projective bundle over its image. Theorem B follows the comutation of Betti numbers of $\mathcal{C}_U^{[n]}$.
- Maulik-Shen-Yin's framework [3] implies
$$C_i H^*(M_{d,\chi}) \subset P_i H^*(M_{d,\chi}), \; * < 2d-4.$$

References

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[3] Davesh Maulik, Junliang Shen, and Qizheng Yin.
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[4] Weite Pi and Junliang Shen.
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