Reading seminar: topics on moduli space of sheaves on surface

Moduli of sheaves on a given variety is a vast subject in algebraic geometry and is related to several branches of mathematics, like enumerative geometry, gauge theory, geometric representation theory etc(see [16] for a comprehensive survey). We plan to understand the cohomology structure of moduli space of sheaves on surfaces. In [6], an interesting conjecture is proposed by Coskun-Woolf that the Betti number $b_i(M_{X,v})$ of moduli space of stable sheaves on algebraic surface X with numerical invariants $v = (r, c_1, c_2) \in H^{2*}(X)$ should stabilise as $\Delta(v) \to \infty$. One of the basic goal of the seminar is to understand the conjecture.

Plan:

1. Sketch the construction of moduli space of sheaves via Gieseker stability [10]. Parallel construction for moduli space of Bridgeland stable object. Modern approach via good moduli theory ([8]) may be discussed. Wu

2. Review the basic properties of moduli space of sheaves on surfaces: e.g., irreducibility [19] [7], especially the elementary transformation (Hecke transform [18]). Singularities of moduli spaces ([10]). Zhang-Zhang

3. The generator of cohomology ring of moduli spaces of sheaves on Possion surfaces following Markan's work [13] [14]. Si

4. Moduli spaces of 1-dimensional sheaves. We focus on topology of the moduli space from enumerative geometric perspective by [3] [5] [4]. We may cover $P = C$ conjecture of [20] [11]. If possible, we try to understand the work of Maulik-Shen-Yin, Bae-Maulik-Shen-Yin on the geometry and topology of Abelian fibrations [15] [1]. Si-Zhang

5. Wall-crossing for the moduli space $M_{X,v}$: e.g [2] for K3 surfaces X, [23] for some elliptic surface. Wall-crossing is a very useful tool to study the geometry and topology of moduli space, which helps us to solve many related problems. Chen-Wu

6. Degeneration methods for moduli of sheaves: moduli of bundles on curve cases [17] [21] and Quot Scheme in [12]. Maybe the related problem for moduli of Higgs bundles will be discussed. Zhang

7. Counting points approach to cohomology. This method relies on Weil conjecture and first application to compute Betti numbers is due to [9] on curves and then [22] on ruled surfaces. Si

Place: Online;   **Time:** Wednesday, 19:00-21:00

Schedule of talks:

- Talk1: Overview of the topics

- Talk2: construction of moduli spaces and cover plane 1.

- $\cdots$

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