# Compactification of moduli space of quasi-polarized K3 surface

Fei Si
Joint work with F. Greer, R. Laza, Zhiyuan Li, Zhiyu Tian

SCMS, Fudan University

14/01/2020

## Table of Contents

# Algebraic curves

Algebraic curve = compact Riemann surface

# Algebraic curves

Algebraic curve = compact Riemann surface

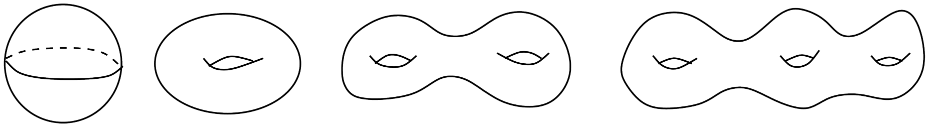


Figure 1: algebraic curves

# Algebraic curves

Algebraic curve = compact Riemann surface

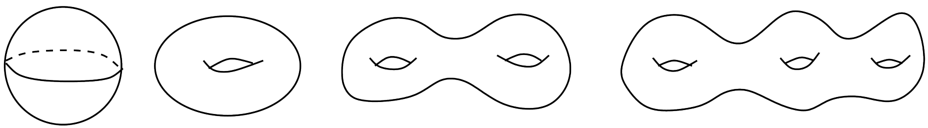


Figure 1: algebraic curves

- $\{\ f(z_0, z_1, z_2) = 0\ \} \subset \mathbb{P}^2$ zeros of degree $4$ polynomial $f$ is genus $3$ curve;

# Algebraic curves

Algebraic curve = compact Riemann surface

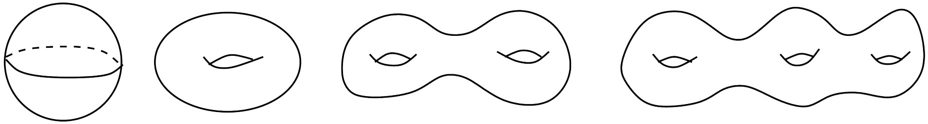


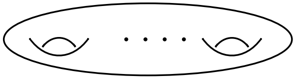
Figure 1: algebraic curves

- $\{\ f(z_0, z_1, z_2) = 0\ \} \subset \mathbb{P}^2$ zeros of degree $4$ polynomial $f$ is genus $3$ curve;
- common zeros of degree $2, 3$ polynomials $\{\ f(z_0, z_1, z_2) = h(z_0, z_1, z_2) = 0\ \} \subset \mathbb{P}^3$ is genus $4$ curve.

# Moduli of curve $M_g$

# Moduli of curve $M_g$

The moduli space $M_g$ of fixed genus $g$ is all isomorphic classes as complex manifolds.

$M_g = \{\Sigma_g :$  $\}/\cong$

# Example: $M_1$

# Example: $M_1$

$g = 1$, elliptic curve

$$M_1 \cong SL(2,\mathbb{Z})\backslash\mathbb{H} \cong SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2)$$

# Example: $M_1$

$g = 1$, elliptic curve

$$M_1 \cong SL(2,\mathbb{Z})\backslash\mathbb{H} \cong SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2)$$

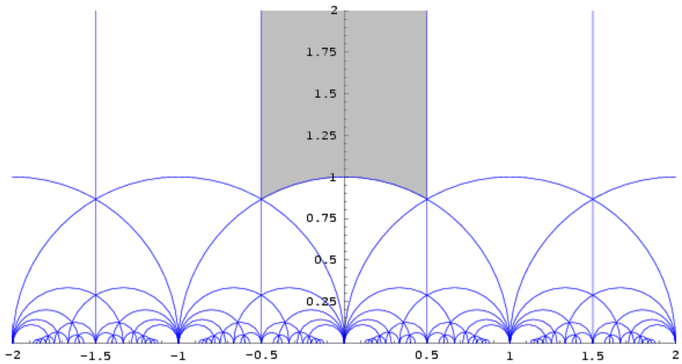



Figure 2: moduli of elliptic curve

# Deligne-Mumford compactification

# Deligne-Mumford compactification

$M_g$ is not compact and not good sometimes, eg, many integrals on $M_g$ from mathematical physics do not make sense.

# Deligne-Mumford compactification

$M_g$ is not compact and not good sometimes, eg, many integrals on $M_g$ from mathematical physics do not make sense.

How to compactify $M_g$ ?

# Deligne-Mumford compactification

$M_g$ is not compact and not good sometimes, eg, many integrals on $M_g$ from mathematical physics do not make sense.

How to compactify $M_g$ ?

In 1970s, Deligne-Mumford compactification $\overline{M}_g$: adding Nodal curves to $M_g$, locally looks like $\{(x,y) \in \mathbb{C}^2 : \ xy = 0 \ \}$

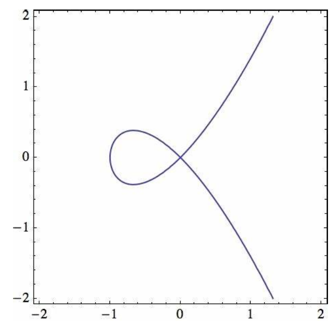



Figure 3: Nodal curve

# Other compactification of $M_g$

# Other compactification of $M_g$

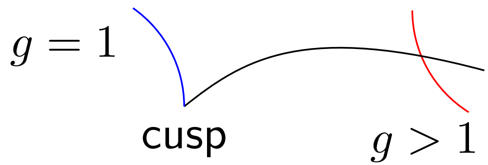
- Mumford-Giesker: : Hilbert scheme and chow variety

# Other compactification of $M_g$

- Mumford-Giesker: : Hilbert scheme and chow variety
- Schubert: Psesudo stable curves, ie, curve admitting elliptic tails with a cusp (worse singularity than nodes).

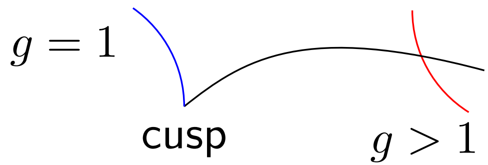
# Other compactification of $M_g$

- Mumford-Giesker: : Hilbert scheme and chow variety
- Schubert: Psesudo stable curves, ie, curve admitting elliptic tails with a cusp (worse singularity than nodes).



# Other compactification of $M_g$

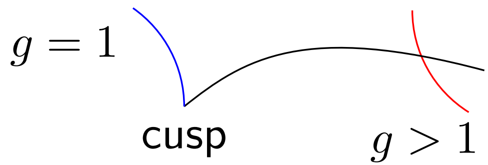
- Mumford-Giesker: : Hilbert scheme and chow variety
- Schubert: Psesudo stable curves, ie, curve admitting elliptic tails with a cusp (worse singularity than nodes).




- Others $\cdots$

# Other compactification of $M_g$

- Mumford-Giesker: : Hilbert scheme and chow variety
- Schubert: Psesudo stable curves, ie, curve admitting elliptic tails with a cusp (worse singularity than nodes).



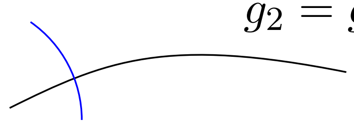

- Others $\cdots$

A systematical way is proposed by Hassett-Hyeon from perspective of birational geometry known as Hassett-Keel Program

# Hassett-Keel Program for $\overline{M}_g$

Divisors:

$\Delta_i := \{$




$\}$

# Hassett-Keel Program for $\overline{M}_g$

Divisors:

$\Delta_i := \{$ 

$\}$

By [BCHM], define log canonical model of $(\overline{M}_g, B)$

$$\overline{M}_g(\beta) := \mathrm{Proj}(\bigoplus_{m \geq 0} H^0(\overline{M}_g, m(K + \beta \cdot B)),\ \beta \in [0,1] \cap \mathbb{Q}$$

where $B := \overline{M}_g - M_g = \Delta_0 + \cdots + \Delta_{\lfloor \frac{g}{2} \rfloor}$ is the boundary divisor.

# First divisorial contraction [B. Hassett, D. Hyeon 2009]

vary $\beta$: wall-crossing

# First divisorial contraction [B. Hassett, D. Hyeon 2009]

vary $\beta$: wall-crossing

$$\begin{array}{c} \overline{M}_g(\frac{9}{11} < \beta \leq 1) \cong \overline{M}_g \cong Hilb_{g,n}/\!\!/SL(N) \\ \downarrow \\ \overline{M}_g(\frac{7}{10} < \beta \leq \frac{9}{11}) \cong \overline{M}_g^{ps} \cong \mathrm{Chow}_3/\!\!/\mathrm{SL}(5\mathrm{g}-5) \end{array}$$

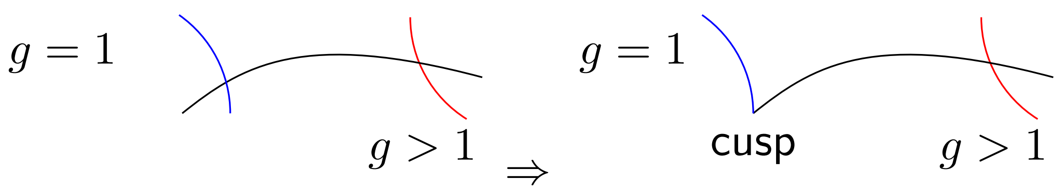
# First divisorial contraction [B. Hassett, D. Hyeon 2009]

vary $\beta$: wall-crossing

$$\overline{M}_g(\tfrac{9}{11} < \beta \leq 1) \cong \overline{M}_g \cong Hilb_{g,n} /\!\!/ SL(N)$$

$$\downarrow$$

$$\overline{M}_g(\tfrac{7}{10} < \beta \leq \tfrac{9}{11}) \cong \overline{M}_g^{ps} \cong \mathrm{Chow}_3 /\!\!/ \mathrm{SL}(5\mathrm{g}-5)$$



# Guiding Problem: Generilise the above to moduli space of quasi-polarised K3

# K3 and example

## Definition

A quasi-polarised K3 surface of genus $g$ is a pair $(S, L)$ where

- $\wedge^2\Omega_S \cong \mathcal{O}_S$ and $\pi_1(S) = 1$.
- $L$: primitive nef line bundle and $L^2 = 2g - 2 > 0$.

# K3 and example

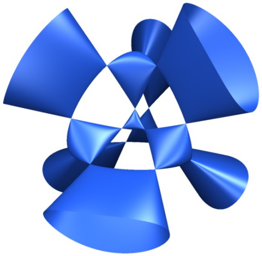
## Definition

A quasi-polarised K3 surface of genus $g$ is a pair $(S, L)$ where

- $\wedge^2\Omega_S \cong \mathcal{O}_S$ and $\pi_1(S) = 1$.
- $L$: primitive nef line bundle and $L^2 = 2g - 2 > 0$.

## Example

complete intersection of cubic and quadric
$S = \{z_0^3 + \cdots z_3^3 = 0,\ q(z_0, .., z_3) = 0\ \} \subset \mathbb{P}^4_{\mathbb{C}}$.



# Moduli space of genus $g$ K3 surfaces $F_g$

Global Torelli: moduli space of quasi-polarised K3 of genus $g$

$$F_g \cong \Gamma_g \backslash SO(2,19)/(SO(2) \times SO(19))$$

is non-compact !

# Moduli space of genus $g$ K3 surfaces $F_g$

Global Torelli: moduli space of quasi-polarised K3 of genus $g$

$$F_g \cong \Gamma_g \backslash SO(2,19)/(SO(2)\times SO(19))$$

is non-compact !

Noether-Lefschetz divisors:

$$\mathcal{D}^g_{d,h} := \{(S,L)\in F_g : \begin{array}{c|c|r} \hline & L & \beta \\ \hline L & 2g-2 & d \\ \hline \beta & d & 2h-2 \\ \hline \end{array} \}$$

# Summaries of Known Compactifications for $F_g$

# Summaries of Known Compactifications for $F_g$

- Arithmetic side: Baily-Borel 1956, Looijenga 2002 $\cdots$

# Summaries of Known Compactifications for $F_g$

- Arithmetic side: Baily-Borel 1956, Looijenga 2002 $\cdots$
- GIT side: $g = 2, 3$ by J. Shah 1980-1984.

# Summaries of Known Compactifications for $F_g$

- Arithmetic side: Baily-Borel 1956, Looijenga 2002 $\cdots$
- GIT side: $g = 2, 3$ by J. Shah 1980-1984.
- Log geometry: Olsson 2004

# Summaries of Known Compactifications for $F_g$

- Arithmetic side: Baily-Borel 1956, Looijenga 2002 $\cdots$
- GIT side: $g = 2, 3$ by J. Shah 1980-1984.
- Log geometry: Olsson 2004
- KSBA: $g = 2$ by R. Laza 2009, ellipptic K3 with marked fiber by Ascher - Bejleri 2018.

# Summaries of Known Compactifications for $F_g$

- Arithmetic side: Baily-Borel 1956, Looijenga 2002 $\cdots$
- GIT side: $g = 2, 3$ by J. Shah 1980-1984.
- Log geometry: Olsson 2004
- KSBA: $g = 2$ by R. Laza 2009, ellipptic K3 with marked fiber by Ascher - Bejleri 2018.
- log $K$-stability : $(\mathbb{P}^2, c \cdot D)$ by Ascher - DeVleming - Liu 2019.

# Summaries of Known Compactifications for $F_g$

- Arithmetic side: Baily-Borel 1956, Looijenga 2002 $\cdots$
- GIT side: $g = 2, 3$ by J. Shah 1980-1984.
- Log geometry: Olsson 2004
- KSBA: $g = 2$ by R. Laza 2009, ellipptic K3 with marked fiber by Ascher - Bejleri 2018.
- log $K$-stability : $(\mathbb{P}^2, c \cdot D)$ by Ascher - DeVleming - Liu 2019.
- Metric side: Gromov-Hausdroff metric by Odaka - Oshima 2019 .

# Summaries of Known Compactifications for $F_g$

- Arithmetic side: Baily-Borel 1956, Looijenga 2002 $\cdots$
- GIT side: $g = 2, 3$ by J. Shah 1980-1984.
- Log geometry: Olsson 2004
- KSBA: $g = 2$ by R. Laza 2009, ellipptic K3 with marked fiber by Ascher - Bejleri 2018.
- log $K$-stability : $(\mathbb{P}^2, c \cdot D)$ by Ascher - DeVleming - Liu 2019.
- Metric side: Gromov-Hausdroff metric by Odaka - Oshima 2019 .
- Tropical geometry: Alexeev - Engel - Thompson 2019.

# Summaries of Known Compactifications for $F_g$

- Arithmetic side: Baily-Borel 1956, Looijenga 2002 $\cdots$
- GIT side: $g = 2, 3$ by J. Shah 1980-1984.
- Log geometry: Olsson 2004
- KSBA: $g = 2$ by R. Laza 2009, ellipptic K3 with marked fiber by Ascher - Bejleri 2018.
- log $K$-stability : $(\mathbb{P}^2, c \cdot D)$ by Ascher - DeVleming - Liu 2019.
- Metric side: Gromov-Hausdroff metric by Odaka - Oshima 2019 .
- Tropical geometry: Alexeev - Engel - Thompson 2019.
- Mirror symmetry: Hacking - Keel - Gross - Siebert 2019.

## Hasset-Keel-Looijenga for $\overline{F_g}^{BB}$

Parallel to $\overline{M}_g$, define

$$\overline{F}_g(a) := \mathrm{Proj}(R(\overline{F_g}^{BB}, K_{F_g} + a \cdot B), \tag{1}$$

where $B$ is a certain combination of Noether-Lefschetz divisors.

# Hasset-Keel-Looijenga for $\overline{F_g}^{BB}$

Parallel to $\overline{M}_g$, define

$$\overline{F}_g(a) := \mathrm{Proj}(R(\overline{F_g}^{BB}, K_{F_g} + a \cdot B), \tag{1}$$

where $B$ is a certain combination of Noether-Lefschetz divisors.

Warning: It is known that $F_g$ is log canonical but not kawamata log terminal.
Current knowledge of MMP can not ensure the finite generation of $R(F_g, K_{F_g} + a \cdot B)$.

# Conjectural Prediction for HKL Program for K3

## Prediction

*The parameter* $a \in \mathbb{Q} \cap [0,1]$ *admits a chamber structure with finite many walls* $0 < a_0 < \cdots < a_m < 1$.

# Conjectural Prediction for HKL Program for K3

## Prediction

*The parameter* $a \in \mathbb{Q} \cap [0,1]$ *admits a chamber structure with finite many walls* $0 < a_0 < \cdots < a_m < 1$.

- *when* $a \in (a_{i-1}, a_i)$, *all* $\overline{F}_g(a)$ *are isomorphic, we denote* $\overline{F}_g(a_{i-1}, a_i)$;

# Conjectural Prediction for HKL Program for K3

## Prediction

*The parameter $a \in \mathbb{Q} \cap [0,1]$ admits a chamber structure with finite many walls $0 < a_0 < \cdots < a_m < 1$.*

- *when $a \in (a_{i-1}, a_i)$, all $\overline{F}_g(a)$ are isomorphic, we denote $\overline{F}_g(a_{i-1}, a_i)$;*
- *when $a$ crosses some wall $a_i$, there is a birational map (typically a flip)*

$$\begin{tikzcd}
\overline{F}_g(a_{i-1}, a_i) \arrow[dr, "f_i^+"] & & \overline{F}_g(a_i, a_{i+1}) \arrow[dl, "f_i^-"'] \\
& \overline{F}_g(a_i) &
\end{tikzcd}$$

# Evidence for the Predictions

$g = 2$, Picard number of $F_2$ is $2$ and Hasset-Keel-Looijenga is easy by J. Shah 1980;

# Evidence for the Predictions

$g = 2$, Picard number of $F_2$ is $2$ and Hasset-Keel-Looijenga is easy by J. Shah 1980;

$g = 3$, there is a series of work K. O'Grady and R. Laza (2016-2018). They give a conjectural picture of the Wall-crossing phenomenon based on some arithmetic consideration and verified partially.

# Evidence for the Predictions

$g = 2$, Picard number of $F_2$ is $2$ and Hasset-Keel-Looijenga is easy by J. Shah 1980;

$g = 3$, there is a series of work K. O'Grady and R. Laza (2016-2018). They give a conjectural picture of the Wall-crossing phenomenon based on some arithmetic consideration and verified partially.
$g = 3$, verified partially via K-stabilty by Ascher - DeVleming - Liu in progress

# Genus $g = 4$ K3 surface and VGIT

- A general quasi-polarized K3 surface $(S, L)$ of genus $4$ K3 surface is a complete intersection of cubics and quadrics in $\mathbb{P}^4$.

# Genus $g = 4$ K3 surface and VGIT

- A general quasi-polarized K3 surface $(S, L)$ of genus $4$ K3 surface is a complete intersection of cubics and quadrics in $\mathbb{P}^4$.
- Parameter space $\pi : \mathbb{P}(E) \to |\mathcal{O}_{\mathbb{P}^4}(2)| \cong \mathbb{P}^{14}$. fiber at $[q]$:

$$E_{[q]} = \{f \in |\mathcal{O}_{\mathbb{P}^4}(3)| : q \text{ is not a factor of } f\}$$

# Genus $g = 4$ K3 surface and VGIT

- A general quasi-polarized K3 surface $(S, L)$ of genus $4$ K3 surface is a complete intersection of cubics and quadrics in $\mathbb{P}^4$.
- Parameter space $\pi : \mathbb{P}(E) \to |\mathcal{O}_{\mathbb{P}^4}(2)| \cong \mathbb{P}^{14}$. fiber at $[q]$:

$$E_{[q]} = \{f \in |\mathcal{O}_{\mathbb{P}^4}(3)| : q \text{ is not a factor of } f\}$$

line bundle $H_t = t \cdot h + \eta$ is ample if and only if $0 < t < \frac{1}{2}$ where $\eta := \pi^* \mathcal{O}_{\mathbb{P}^{14}}(1)$ and $h := \mathcal{O}_{\mathbb{P}(E)}(1)$

# Genus $g = 4$ K3 surface and VGIT

- A general quasi-polarized K3 surface $(S, L)$ of genus $4$ K3 surface is a complete intersection of cubics and quadrics in $\mathbb{P}^4$.
- Parameter space $\pi : \mathbb{P}(E) \to |\mathcal{O}_{\mathbb{P}^4}(2)| \cong \mathbb{P}^{14}$. fiber at $[q]$:

$$E_{[q]} = \{f \in |\mathcal{O}_{\mathbb{P}^4}(3)| : q \text{ is not a factor of } f\}$$

  line bundle $H_t = t \cdot h + \eta$ is ample if and only if $0 < t < \frac{1}{2}$ where $\eta := \pi^* \mathcal{O}_{\mathbb{P}^{14}}(1)$ and $h := \mathcal{O}_{\mathbb{P}(E)}(1)$

define Variation of Geometric invariant theory (VGIT)

$$\mathfrak{M}(t) := \mathbb{P}E /\!\!/_{H_t} SL(5), \quad t > 0$$

# Other parameter spaces

- $|\mathcal{O}_Q(3)|$: sections space on a fixed smooth quadric $3$-fold;

# Other parameter spaces

- $|\mathcal{O}_Q(3)|$: sections space on a fixed smooth quadric $3$-fold;
- $\mathrm{Chow}$: the main irreducible component of chow variety parametrizing $2$-cycles in $\mathbb{P}^4$ of degree $6$;

# Other parameter spaces

- $|\mathcal{O}_Q(3)|$: sections space on a fixed smooth quadric $3$-fold;
- $\mathrm{Chow}$: the main irreducible component of chow variety parametrizing $2$-cycles in $\mathbb{P}^4$ of degree $6$;
- $\Delta_0$ : Hilbert scheme of singular cubic $4$-folds.

## Theorem (Greer-Laza-Li-Si-Tian, 2019)

- *we have identification*
  - $0 < t < \frac{1}{6}$, $\mathfrak{M}(t) \cong |\mathcal{O}_Q(3)| /\!\!/ SO(5)$ .
  - $t = \frac{2}{3}$, $\mathfrak{M}(t) \cong \mathrm{Chow} /\!\!/ \mathrm{SL}(5)$
  - $t = 1$, $\mathfrak{M}(t) \cong (\Delta_0 /\!\!/ SL(6))^{\nu}$

## Theorem (Greer-Laza-Li-Si-Tian, 2019)

- *we have identification*
  - $0 < t < \frac{1}{6}$, $\mathfrak{M}(t) \cong |\mathcal{O}_Q(3)| /\!\!/ SO(5)$ .
  - $t = \frac{2}{3}$, $\mathfrak{M}(t) \cong \mathrm{Chow} /\!\!/ \mathrm{SL}(5)$
  - $t = 1$, $\mathfrak{M}(t) \cong (\Delta_0 /\!\!/ SL(6))^\nu$
- *The walls of VGIT for* $0 < t < 1$ *are the following*

$$\left\{\frac{1}{6}, \frac{2}{7}, \frac{3}{8}, \frac{4}{9}, \frac{1}{2}, \frac{6}{11}, \frac{7}{12}, \frac{8}{13}, \frac{2}{3}\right\}$$

# Wall-Crossing

first wall-crossing: divisorial contraction

# Wall-Crossing

first wall-crossing: divisorial contraction

$$\begin{tikzcd} & \mathfrak{M}(\frac{1}{6}, \frac{2}{7}) \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ \mathfrak{M}(0, \frac{1}{6}] & & \mathfrak{M}(\frac{2}{7}) \end{tikzcd}$$

# HKL for genus $4$

We choose $B$ such that

$$K + a \cdot B = \lambda + s \cdot (\mathcal{D}_{0,0} + \mathcal{D}_{1,1} + \mathcal{D}_{2,1} + \mathcal{D}_{3,1})$$

# HKL for genus $4$

We choose $B$ such that

$$K + a \cdot B = \lambda + s \cdot (\mathcal{D}_{0,0} + \mathcal{D}_{1,1} + \mathcal{D}_{2,1} + \mathcal{D}_{3,1})$$

then we can show

**Theorem (Greer-Laza-Li-Si-Tian)**

$$\mathfrak{M}(t) \cong \overline{F}_4(\beta(t))$$

*for certain rational function $\beta(t)$.*

# Thanks for your attention !